Decentralized Mean Orbit Element Formation Stability for Uncoordinated Maneuvers

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***Abstract*—Sufficient stability conditions are derived for simultaneous uncoordinated impulsive maneuvers in distributed mean orbit element spacecraft formations. Lyapunov stability formalisms are used in conjunction with distributed mean orbit element spacecraft formation definitions. Special cases of the sufficient stability conditions are examined. Simulated results demonstrate the efficacy of the approach, and conclusions and future work are discussed.**

I. INTRODUCTION

On-orbit Formation Flight (FF) Guidance, Navigation, & Control (GNC) has received significant attention over the last decade. The need for solutions to this problem has become more defined in recent years due to emerging mission concepts such as fractionation [1]. The fractionation approach is predicated on functionally decomposing large spacecraft into smaller, specialized spacecraft, drastically increasing the system complexity of on-board autonomous fault management systems. Ground station operational burdens combined with large gaps in uplink/downlink connectivity mean then centralized ground control of the entire formation is infeasible. On-orbit centralized control of the formation conditions the safety of the entire formation on the continued operation of a single spacecraft, incentivizing decentralized GNC implementations.

The on-orbit dynamic environment subjects individual spacecraft to significant oscillating, gross, and differential perturbations. Mean orbit elements effectively 'average out' oscillating disturbances and are valid for all orbit regimes [2], [3]. This makes differential mean orbit elements particularly useful for specifying relative spacecraft geometry. Gross perturbations affect the entire formation, and can incur substantial Δv costs if they are rejected. Rather, it is preferable for long-term orbit maintenance that gross perturbations be ignored in the short term and relative formation maintenance enforced. Differential mean orbit elements relative to a weighted formation barycenter are used for a number of reasons. First, unperturbed differential elements are constants of motion for arbitrary Keplerian orbit regimes (depending of course on the orbit elements chosen) and as such do not change with time, making formation 'slot' definitions easy and intuitive. Second, differential motion can be examined under the effects of perturbations (e.g., Earth oblateness) and partially mitigated using intelligent differential mean orbit element choices [4], [5], [6]. Third, because differential mean orbit elements change slowly under perturbations, they are ideal consensus variables.

The challenge facing on-orbit formation guidance, navigation, and control, in a decentralized fashion, is 1) for the individual spacecraft to agree on where the perturbed formation barycenter is located, 2) to know their desired relative state to achieve formation objectives, and 3) actuate to reduce relative state error while minimizing Δv (fuel). This paper focuses on deriving admissible maneuver constraints on uncoordinated individual or simultaneous impulsive maneuvers.

Recent advances in cooperative control are immediately useful in addressing this challenge. The problem of spacecraft in a formation with intermittent communications agreeing on the location of the moving barycenter may be directly framed as a consensus problem over random directed graphs [7], [8], [9], [10]. As each spacecraft is assumed to have its own on-board estimation capabilities (e.g. Extended Kalman Filters or Unscented Kalman Filters), the consensus problem needs to leverage provided estimation knowledge to quantify the uncertainty of the consensus barycenter (in a fashion similar to Distributed Kalman Filters [11] or Distributed Bayesian Estimation[12]). Distributed spacecraft formation flight has been previously investigated using graph theory [13], [14].

This paper derives sufficient stability conditions on simultaneous uncoordinated impulsive control laws for the decentralized mean orbit element formation flight GNC approach outlined in previous work [15], [16]. Special cases for sufficient stability conditions of asynchronous uncoordinated impulsive control are also examined. Analytical predictions are verified using simulations. Conclusions and future work are discussed.

II. BACKGROUND

Before detailing the contributions of this paper, it is necessary to introduce the formalisms developed in previous work [15] to define the distributed spacecraft formation, barycenter, and individual formation 'slots.'

Definition 1: Weighted Formation Barycenter
The formation mean orbit element barycenter $\bar{\text{œ}}^b_k$ at time t_k given spacecraft weights w^i_k and reference differential mean orbit elements $\delta\bar{\text{œ}}^i_{r,k}$ is defined as

$$\bar{\text{œ}}^b_k = \sum_{i=1}^{N_f} w^i_k \bar{\text{œ}}^{b,i}_k = \sum_{i=1}^{N_f} w^i_k \left(\bar{\text{œ}}^i_k - \delta\bar{\text{œ}}^i_{r,k}\right) \qquad (1)$$

where the weights $w_k^i \in \mathbb{R}$ are such that $0 \leq w_k^i \leq 1$, $i = 1, \ldots, N_f$, and $w_k^1 + \cdots + w_k^{N_f} = 1$.

The differential mean orbit element reference offset (or formation 'slot') $\delta\bar{\text{œ}}_{r,k}^i$ may also be defined implicitly given a weighted formation barycenter $\bar{\text{œ}}_k^b$ and instantaneous reference mean orbit element $\bar{\text{œ}}_{r,k}^i$.

$$\delta\bar{\text{œ}}_{r,k}^i = \bar{\text{œ}}_{r,k}^i - \bar{\text{œ}}_k^b \tag{2}$$

Note that $\bar{\text{œ}}_{r,k}^i$ is the reference mean orbit elements, not the instantaneous orbit elements $\bar{\text{œ}}_k^i$.

Definition 2: Formation Slot
A formation slot is defined by a spacecraft's choice of $\delta\bar{\text{œ}}_{r,k}^i$. The only constraints placed on $\delta\bar{\text{œ}}_{r,k}^i$ is that they be well defined on the orbit element space (e.g., differential mean eccentricity $\bar{e}_k^i$ such that $0 \leq e_k^b + \delta\bar{e}_k^i < 1$) and satisfy user-defined constraints, such as collision avoidance and other operational needs.

Note, specific 'user-defined constraints' are introduced and discussed in McMahon & Holzinger [16], however in summary they include invariance to gravitational perturbations and constraints due to collision safety. The definition for a relative formation is now given.

Definition 3: Relative Formation
A spacecraft formation is said to be a relative formation when each spacecraft is aware of all other formation spacecraft (i.e. given a set of formation spacecraft $\mathcal{F}$, the size of $\mathcal{F}$, N_f, is known, and a specific spacecraft is associated with each $i \in \mathcal{F}$), their respective current mean orbit elements $\bar{\text{œ}}_k^i$, and barycenter weightings w_k^i. Further, each spacecraft must know its own differential mean orbit element formation slot $\delta\bar{\text{œ}}_{r,k}^i$.

The instantaneous value of the differential mean orbit elements is given by

$$\delta\bar{\text{œ}}_k^i = \bar{\text{œ}}_k^i - \bar{\text{œ}}_k^b \tag{3}$$

Explicitly, the instantaneous differential mean orbit element error is defined as

$$\delta\bar{\mathbf{e}}_k^i = \delta\bar{\text{œ}}_{r,k}^i - \delta\bar{\text{œ}}_k^i \tag{4}$$

Necessarily, when $\delta\bar{\mathbf{e}}_k^i \rightarrow \mathbf{0}\ \forall i \in \mathcal{F}$, the formation has exactly zero error. The following section contains the primary contributions of this paper, wherein sufficient conditions on changes in spacecraft mean orbit elements due to impulsive maneuvers are developed and discussed.

III. Stability Analysis

General stability analysis results are first discussed, followed by special cases for uncoordinated maneuvers by single or multiple spacecraft in formations.

A. General Results

The following Definitions, Lemmas, and Corollaries identify sufficient conditions for formation stability, then continue to discuss consequences of these results. As a matter of notational convenience, the time subscript $(\cdot)_k$ is dropped and considered to be implicit for the remainder of the theoretical derivation. First, the Lyapunov function used to measure stability is given in Definition 4.

Definition 4: Formation Error Lyapunov Function
The formation error Lyapunov function is defined as

$$V = \frac{1}{2}\sum_{j\in\mathcal{F}}\left(\delta\bar{\mathbf{e}}^j \cdot \delta\bar{\mathbf{e}}^j\right) \tag{5}$$

The mean orbit element state error for each spacecraft j in the formation $\mathcal{F}$ is weighted equally. Note the definition of V in (5) is such that $V \geq 0$ for all $\delta\bar{\mathbf{e}}^j$, $j \in \mathcal{F}$.

An impulsive maneuver must necessarily produce an instantaneous change in a spacecraft mean orbit elements. However, because impulses in 3-DoF Newtonian systems are restricted to a 3 dimensional subspace of $\mathbb{R}^6$, They may only instantaneously change a 3 dimensional subspace of the mean orbit elements of a spacecraft. This is notationally captured as

$$\Delta\bar{\text{œ}}_k^i = \mathbf{B}(\bar{\text{œ}}_k^i)\Delta\mathbf{v}_k^i \tag{6}$$

with $\mathbf{B}(\bar{\text{œ}}_k^i) \in \mathbb{R}^{6\times 3}$, which allows the mean orbit elements both immediately before and after an impulsive maneuver to be written as

$$\bar{\text{œ}}_k^{i,+} = \bar{\text{œ}}_k^i + \Delta\bar{\text{œ}}_k^i = \bar{\text{œ}}_k^i + \mathbf{B}(\bar{\text{œ}}_k^i)\Delta\mathbf{v}_k^i$$

Now that the Lyapunov function has been defined, the sufficient condition for formation stability when a single spacecraft maneuvers impulsively is given in Lemma 1.

Lemma 1: Impulsive Formation Control Stability Sufficient Condition
Given (*i*) a formation $\mathcal{F}$ with N_f spacecraft, (*ii*) a set of N_m simultaneously maneuvering spacecraft $\mathcal{M} \subseteq \mathcal{F}$, $\mathcal{M} \neq \emptyset$, (*iii*) a set of N_q quiescent spacecraft $\mathcal{Q} \subset \mathcal{F}$, where $\mathcal{M} \cup \mathcal{Q} = \mathcal{F}$ and $\mathcal{M} \cap \mathcal{Q} = \emptyset$, and (*iv*) if all weights w^j and Formation Slots $\delta\bar{\text{œ}}_r^j$ are known and agreed upon by the formation, a sufficient condition for formation stability under impulsive maneuvers for each satellite i maneuvering at any time-step k is

$$\sum_{i\in\mathcal{M}}\left\{\frac{1}{2}k(w^i, N_f)\Delta\bar{\text{œ}}^i \cdot \Delta\bar{\text{œ}}^i - \left(\delta\bar{\mathbf{e}}^i - w^i\sum_{j\in\mathcal{F}}\delta\bar{\mathbf{e}}^j\right)\cdot\Delta\bar{\text{œ}}^i + w^i\left[\sum_{j\neq i\in\mathcal{M}}c(w^j, N_f)\Delta\bar{\text{œ}}^j\right]\cdot\Delta\bar{\text{œ}}^i\right\} \leq 0 \tag{7}$$

where

$$k(w^i, N_f) = 1 - 2w^i + N_f(w^i)^2 \tag{8}$$

and

$$c(w^j, N_f) = \frac{N_f}{2} w^j - 1 \tag{9}$$

Proof: To demonstrate stability, the Lyapunov function defined in (5) both before (V) and after (V^+) the impulse is used.

$$V = \frac{1}{2}\sum_{i\in\mathcal{M}} \delta\bar{\mathbf{e}}^i \cdot \delta\bar{\mathbf{e}}^i + \frac{1}{2}\sum_{j\in\mathcal{Q}} \delta\bar{\mathbf{e}}^j \cdot \delta\bar{\mathbf{e}}^j$$

$$V^+ = \frac{1}{2}\sum_{i\in\mathcal{M}} \delta\bar{\mathbf{e}}^{i,+} \cdot \delta\bar{\mathbf{e}}^{i,+} + \frac{1}{2}\sum_{j\in\mathcal{Q}} \delta\bar{\mathbf{e}}^{j,+} \cdot \delta\bar{\mathbf{e}}^{j,+}$$

The relationship between $\delta\bar{\mathbf{e}}$, $\delta\bar{\mathbf{e}}^+$, and $\Delta\bar{\mathbf{œ}}$ can be found using the definition of the formation barycenter in Eq. (1) and the instantaneous differential mean orbit element error in Eq. (4) for both maneuvering and quiescent spacecraft. For maneuvering spacecraft it can be shown that,

$$\delta\bar{\mathbf{e}}^{i,+} = \delta\bar{\mathbf{e}}^i - \Delta\bar{\mathbf{œ}}^i + \sum_{p\in\mathcal{M}} w^p \Delta\bar{\mathbf{œ}}^p$$

Similarly, for the quiescent spacecraft a similar relation can be formulated:

$$\delta\bar{\mathbf{e}}^{j,+} = \delta\bar{\mathbf{e}}^j + \sum_{p\in\mathcal{M}} w^p \Delta\bar{\mathbf{œ}}^p$$

Substituting $\delta\bar{\mathbf{e}}^{j,+}$ and $\delta\bar{\mathbf{e}}^{j,+}$ into the relation for V^+, computing $V^+ - V \leq 0$, and engaging in significant algebraic manipulation, the following form can be obtained:

$$\begin{aligned} V^+ - V = \sum_{i\in\mathcal{M}} \Bigg\{ &\frac{1}{2}\left(1 - 2w^i + N_f (w^i)^2\right) \Delta\bar{\mathbf{œ}}^i \cdot \Delta\bar{\mathbf{œ}}^i \\ &- \left(\delta\bar{\mathbf{e}}^i - w^i \sum_{j\in\mathcal{F}} \delta\bar{\mathbf{e}}^j\right) \cdot \Delta\bar{\mathbf{œ}}^i \\ &+ w^i \left[\sum_{j\neq i\in\mathcal{M}} \left(w^j \frac{N_f}{2} - 1\right) \Delta\bar{\mathbf{œ}}^j\right] \cdot \Delta\bar{\mathbf{œ}}^i \Bigg\} \end{aligned}$$

For the Lyapunov error function (5) to strictly decrease, it is sufficient to require that $V^+ - V \leq 0$. Using the definition of $k(w^i, N_f)$ given in (8) and $c(w^j, N_f)$ given in (9), the final form of the sufficient condition for Lyapunov stability shown in (7) is obtained.
□

Remark 1: Control Law Agnosticism
From inspection of the sufficiency condition in Lemma 1 for stability with multiple simultaneous impulsive maneuvers in a distributed mean orbit element spacecraft formation, it can be seen that (7) is independent of any control policy used by individual spacecraft. Rather, (7) should be considered a conservative constraint on control policies implemented on individual spacecraft in scenarios where formation stability is desirable.

The form of (7) is, in general, a quadratic inequality with respect to $\Delta\bar{\mathbf{œ}}^i$ and $\Delta\bar{\mathbf{œ}}^j$. A matrix representation of (7) provides insight on how admissible maneuvers may be generated. Let

$$\mathbf{z}^T = \begin{bmatrix} (\Delta\bar{\mathbf{œ}}^1)^T & \cdots & (\Delta\bar{\mathbf{œ}}^i)^T & \cdots & (\Delta\bar{\mathbf{œ}}^{N_m})^T \end{bmatrix}$$

$$\mathbf{A}_{ij} = \begin{cases} k(w^i, N_f)\mathbb{I} & \text{if } j = i \\ 2w^i c(w^j, N_f)\mathbb{I} & \text{if } j \neq i \end{cases}$$

$$\mathbf{A} = \begin{bmatrix} & & \vdots & & & & \\ \mathbf{A}_{i1} & \cdots & \mathbf{A}_{ii} & \cdots & \mathbf{A}_{ij} & \cdots & \mathbf{A}_{i,N_m} \\ & & \vdots & & & & \end{bmatrix}$$

$$\mathbf{b}_i = \delta\bar{\mathbf{e}}^i - w^i \sum_{j\in\mathcal{F}} \delta\bar{\mathbf{e}}^j$$

Then, (7) may be written as

$$\frac{1}{2}\mathbf{z}^T \mathbf{A}\mathbf{z} - \mathbf{b}^T \mathbf{z} \leq 0 \tag{10}$$

As a trivial solution, (7) admits $\mathbf{z} = \mathbf{0}$ (e.g., maneuvers of zero magnitude).

Corollary 1: Min. Lyapunov Error Solution Existence
A minimum unique Lyapunov error maneuver solution to (7) always exists for a formation $\mathcal{F}$ containing N_f spacecraft with maneuvering spacecraft $i \in \mathcal{M}$, $w_i \in [0,1]$, $\sum_{i\in\mathcal{F}} w^i = 1$. Further, the minimum Lyapunov error solution can be described by

$$\mathbf{z}^* = \mathbf{A}^{-1}\mathbf{b} \tag{11}$$

Proof: The first and second-order necessary conditions of optimality provide a solution to (7) in the form of (11), provided that $\mathbf{A}^{-1}$ exists. The existence of $\mathbf{A}^{-1}$ is shown by demonstrating that every row in $\mathbf{A}$ for arbitrary spacecraft $i \neq j$ are linearly independent. The approach here demonstrates this using proof by contradiction. The row blocks corresponding to the i^{th} and j^{th} spacecraft may be written as

$$\mathbf{A}_i = \begin{bmatrix} \cdots & \mathbf{A}_{ii} & \cdots & \mathbf{A}_{ij} & \cdots & \mathbf{A}_{ik} & \cdots \end{bmatrix}$$
$$\mathbf{A}_j = \begin{bmatrix} \cdots & \mathbf{A}_{ji} & \cdots & \mathbf{A}_{jj} & \cdots & \mathbf{A}_{jk} & \cdots \end{bmatrix}$$

Because each $\mathbf{A}_{(\cdot,\cdot)}$ is a scalar multiplied by an identity matrix, if $\mathbf{A}_i$ is linearly dependent on $\mathbf{A}_j$, it is simply required that

$$k(w^i, N_f) = \alpha 2w^j c(w^i, N_f)$$
$$2w^i c(w^j, N_f) = \alpha k(w^j, N_f)$$
$$2w^i c(w^k, N_f) = \alpha 2w^j c(w^k, N_f)$$

here, $k \in \mathcal{F}$, $k \neq i, j$. It can be seen that for the final equality to be true, $\alpha = w^i/w^j$. Substuting this relationship for α in to the first equation reduces to $w^j = 1$. Then, with $\alpha = w^i/w^j$ and $w^j = 1$, the second equation requires that $w^i = \sqrt{1/N_f}$. However, this is not possible, because $\sum_{i\in\mathcal{F}} w^i = 1$ where $w^i \in [0,1]$. Thus, the rows cannot possibly be linearly dependant on one another and must therefor be linearly independent. If this is the case for any arbitrary combination of rows, then all the rows of $\mathbf{A}$ are linearly independent, and $\mathbf{A}^{-1}$ exists. Thus, a unique $\mathbf{z}^*$ always exists

□

Corollary 2: Stationary Barycenter Simultaneous Maneuver Stability

Under the same conditions as Corollary 1, the unique minimum Lyapunov error solution to (7) with the additional constraint that the formation barycenter (1) remain stationary is

$$\mathbf{z} = \mathbf{A}^{-1}\left[\bar{\mathbf{b}} + \left(\mathbf{W}\mathbf{W}^T\right)^{-1}\mathbf{W}\mathbf{A}^{-1}\bar{\mathbf{b}}\right] \tag{12}$$

where

$$\mathbf{W} = \begin{bmatrix} w^1\mathbb{I} & \cdots & w^i\mathbb{I} & \cdots & w^{N_m}\mathbb{I} \end{bmatrix}^T$$

captures the weights of the simultaneously maneuvering spacecraft.

Proof: The barycenter location immediately after impulsive maneuvers by spacecraft in $\mathcal{M}$ can be written as

$$\bar{\text{œ}}^{b,+} = \sum_{i\in\mathcal{M}} w^i\left(\bar{\text{œ}}^{i,+} - \delta\bar{\text{œ}}^i_r\right) + \sum_{j\in\mathcal{Q}} w^i\left(\bar{\text{œ}}^j - \delta\bar{\text{œ}}^j_r\right)$$

For each maneuvering spacecraft in $\mathcal{M}$, $\bar{\text{œ}}^{i,+} = \bar{\text{œ}}^i + \Delta\bar{\text{œ}}^i$. After some manipulation, it can be seen that

$$\sum_{i\in\mathcal{M}} w^i\Delta\bar{\text{œ}}^i = \mathbf{W}\mathbf{z} = \mathbf{0}$$

The result in (12) is then found by applying the first- and second-order necessary conditions of optimality to the augmented Lyapunov error decrease function.

□

Until now, it has been tacitly assumed that all maneuver information is available to each spacecraft (i.e., spacecraft i is aware of all simultaneous maneuvers $\Delta\bar{\text{œ}}^j$, $j \in \mathcal{M}$. Next, necessary formation information sharing and uncoordinated maneuvers are discussed.

Remark 2: Formation Spacecraft Knowledge Requirements

Further examination of the information necessary to satisfy (7) reveals that, for each maneuvering spacecraft $i \in \mathcal{M}$, all differential mean orbit element errors $\delta\bar{\mathbf{e}}^j$, $j \in \mathcal{F}$ must be known. Additionally, it is helpful (though not strictly necessary) if i^{th} spacecraft is aware of both the members of the set $\mathcal{M}$ as well as their respective maneuvers $\Delta\bar{\text{œ}}^j$.

Definition 5: Uncoordinated Maneuvers

A formation $\mathcal{F}$ with spacecraft maneuvering (individually or simultaneously) without sharing information about its/their own individual maneuvers is defined to be executing uncoordinated maneuvers. The assumption is made that no spacecraft in $\mathcal{F}$ has any information about any maneuvers executed by other spacecraft. Thus, in the absence of sharing, for each spacecraft, it must assumed that $\mathcal{M} \to \mathcal{F}$. Under both of these assumptions, it is logically the responsibility of each spacecraft to ensure that their corresponding term in (7) is individually satisfied (≤ 0).

In the event that no knowledge of the simultaneous j^{th} spacecraft maneuver $\Delta\bar{\text{œ}}^j$ or the membership of $\mathcal{M}$ exists, the question that is now addressed is how the i^{th} spacecraft ($\forall i \in \mathcal{F}$) must maneuver, should it choose to do so, to ensure formation error Lyapunov stability. Before discussing special cases for multiple and single impulse formation maneuvers, the definitions for leader / follower and democratic formations are given.

Definition 6: Leader / Follower Formation

A formation $\mathcal{F}$ is a leader / follower formation if and only if $w^i = 1$ (designating the i^{th} spacecraft as the leader) and all other $j \neq i \in \mathcal{F}$ have weightings such that $w^j = 0$.

Definition 7: Democratic Formation

A formation $\mathcal{F}$ is a democratic formation if and only if for all $i \in \mathcal{F}$, $w^i = 1/N_f$, where N_f is the number of spacecraft in $\mathcal{F}$.

B. Single-Impulse Simplifications

For a leader / follower formation, the following single-spacecraft uncoordinated maneuver sufficient stability condition is developed here.

Corollary 3: Single-impulse maximum formation error reduction

The mean orbit element state change resulting from a maneuver that maximally decreases the formation state error is

$$\Delta\bar{\text{œ}}^{i,*} = \frac{1}{1 - 2w^i + N_f(w^i)^2}\left(\delta\bar{\mathbf{e}}^i - w^i\left[\sum_{j\in\mathcal{F}}\delta\bar{\mathbf{e}}^j\right]\right) \tag{13}$$

Proof: This is seen by simplifying (11) in Corollary 1.

□

Again, it must be emphasized that a null maneuver (e.g., remaining stationary) also satisfies the sufficient condition. As $w^i \to 0$ (a 'follower' spacecraft),

$$\Delta\bar{\text{œ}}^{i,*}\big|_{w^i\to 0} = \delta\bar{\mathbf{e}}^i$$

In this case $w^i \to 0$, the state change $\Delta\bar{\text{œ}}^i$ does not effect the state error of any other spacecraft in the formation, and thus the optimal policy is to simply remove the state error for the i^{th} spacecraft. As $w^i \to 1$ (a 'leader' spacecraft),

$$\Delta\bar{\text{œ}}^{i,*}\big|_{w^i\to 1} = -\frac{1}{N_f - 1}\left[\sum_{j\neq i\in\mathcal{F}}\delta\bar{\mathbf{e}}^j\right]$$

Importantly, the i^{th} spacecraft error no longer appears in the equation, and the policy for the i^{th} spacecraft (with $w^i = 1$) that maximally decreases the formation state error is to move such that the equally-weighted mean spacecraft error for all follower spacecraft is zero. Note that, for a leader spacecraft, $\Delta\bar{\text{œ}}^i = \mathbf{0}$ is also admissible, per Lemma 1.

Examining (13) for a Democratic formation as $w^i \to 1/N_f$ produces

$$\Delta\bar{\text{œ}}^{i,*}\big|_{w^i \to \frac{1}{N_f}} = \frac{N_f}{N_f - 1}\left(\delta\bar{\mathbf{e}}^i - \frac{1}{N_f}\left[\sum_{j\in\mathcal{F}}\delta\bar{\mathbf{e}}^j\right]\right)$$

Corollary 4 now describes admissible $\Delta\bar{\text{œ}}^i$ for each non-simultaneous uncoordinated maneuver of spacecraft $i \in \mathcal{F}$.

Corollary 4: Spherical Admissible Regions for Individual Stable Maneuvers

Under the same assumptions as Lemma 1, but with $\mathcal{M}$ containing only the i^{th} spacecraft, any maneuver resulting in a state change $\Delta\bar{\text{œ}}^i$ that may be parameterized as

$$\Delta\bar{\text{œ}}^i = \Delta\bar{\text{œ}}^{i,*} + \varepsilon^i\|\Delta\bar{\text{œ}}^{i,*}\|\hat{\mathbf{r}}^i \tag{14}$$

where $\varepsilon^i \in [0,1]$ and $\hat{\mathbf{r}}^i$ is any unit vector, is a maneuver that satisfies (7) in Lemma 1, and is therefore a formation stabilizing maneuver.

Proof: This is seen by substituting (14) into (7):

$$\frac{k(w^i, N_f)}{2}\Delta\bar{\text{œ}}^i \cdot \Delta\bar{\text{œ}}^i - \left(\delta\bar{\mathbf{e}}^i - w^i\left[\sum_{j=1}^{N_f}\delta\bar{\mathbf{e}}^j\right]\right)\cdot\Delta\bar{\text{œ}}^i \leq 0$$

Which can be rearranged such that

$$\frac{1}{2}\Delta\bar{\text{œ}}^{i,*}\cdot\Delta\bar{\text{œ}}^{i,*} + \varepsilon^i\|\Delta\bar{\text{œ}}^{i,*}\|\Delta\bar{\text{œ}}^{i,*}\cdot\hat{\mathbf{r}}^i + \frac{1}{2}(\varepsilon^i)^2\|\Delta\bar{\text{œ}}^{i,*}\|^2 \\ -\Delta\bar{\text{œ}}^{i,*}\cdot\left(\Delta\bar{\text{œ}}^{i,*} + \varepsilon^i\|\Delta\bar{\text{œ}}^{i,*}\|\hat{\mathbf{r}}^i\right) \leq 0$$

which, after further simplification, reduces to (14)
□

It is important to note that Eq. (14) allows for admissible maneuvers which are not required to arbitrarily modify an orbit element set, which is crucial due to the previous issue raised concerning an arbitrary $\Delta\mathbf{v}^i$ only affecting a subspace of the mean orbit element space. This result implies that there are a range of admissible $\Delta\mathbf{v}^i$ maneuvers that are stabilizing.

C. Multiple-Impulse Simplifications

The following results apply when any number of spacecraft in $\mathcal{F}$ may be maneuvering, but are not aware of other maneuvers being simultaneously executed by other spacecraft.

Corollary 5: Leader / Follower Admissible Uncoordinated Maneuvers

For a leader / follower formation $\mathcal{F}$ (per Definition 6) with all spacecraft potentially conducting uncoordinated maneuvers (per Definition 5), admissible $\Delta\bar{\text{œ}}^i$ and $\Delta\bar{\text{œ}}^j$ maneuvers (where spacecraft i is the leader and $j \neq i \in \mathcal{F}$ are the follower spacecraft) are

$$\Delta\bar{\text{œ}}^i = \mathbf{0} \tag{15}$$

$$\Delta\bar{\text{œ}}^j = \delta\bar{\mathbf{e}}^j + \varepsilon^j\|\delta\bar{\mathbf{e}}^j\|\hat{\mathbf{r}}^j \tag{16}$$

Where ε^j and $\hat{\mathbf{r}}^j$ are defined in Corollary 4.

Proof: Under Definition 6, when $w^i = 1$ and $w^j = 0\ \forall j \neq i$, (7) reduces relationships for the leader and the followers. For each of the j followers ($w^j = 0$), using Corollaries 3 and 4 it is straightforward to show that (7) is satisfied for the j^{th} spacecraft when

$$\Delta\bar{\text{œ}}^j = \delta\bar{\mathbf{e}}^j + \varepsilon^j\|\delta\bar{\mathbf{e}}^j\|\hat{\mathbf{r}}^j$$

Note that $\Delta\bar{\text{œ}}^j = \mathbf{0}$ is admissible. Using this solution for $\Delta\bar{\text{œ}}^j$, the summation term in (7) spacecraft is

$$\frac{N_f - 1}{2}\Delta\bar{\text{œ}}^i\cdot\Delta\bar{\text{œ}}^i + \sum_{j\neq i\in\mathcal{F}}\delta\bar{\mathbf{e}}^j\cdot\Delta\bar{\text{œ}}^i \\ - \sum_{j\neq i\in\mathcal{F}}\left(\bar{\mathbf{e}}^j + \varepsilon^j\|\delta\bar{\mathbf{e}}^j\|\hat{\mathbf{r}}^j\right)\cdot\Delta\bar{\text{œ}}^i \leq 0$$

Realizing that the summation over $\delta\bar{\mathbf{e}}^j$ cancels and $\hat{\mathbf{r}}^j\cdot\Delta\bar{\text{œ}}^i = \pm\|\Delta\bar{\text{œ}}^i\|$, the sufficient condition for the leader spacecraft then becomes

$$\frac{N_f - 1}{2}\|\Delta\bar{\text{œ}}^i\|^2 \mp \sum_{j\neq i\in\mathcal{F}}\left(\varepsilon^j\|\delta\bar{\mathbf{e}}^j\|\|\Delta\bar{\text{œ}}^i\|\right) \leq 0$$

Which, because the sign of the second term cannot be known without sharing information with the follower spacecraft, requires that $\|\Delta\bar{\text{œ}}^i\| = 0$, producing $\Delta\bar{\text{œ}}^i = \mathbf{0}$. □

Thus, Corollary 5 has demonstrated that in the absence of planned maneuver sharing, it is necessary that the leader not maneuver ($\Delta\bar{\text{œ}}^i = \mathbf{0}$), while the follower spacecraft are free to maneuver as described in (16). Selected results developed in the Theory section are now examined using simulation.

IV. Simulation & Results

The central results in the of this paper are verified here using simulation. Primarily, the formation stability sufficient condition (7) proven in Lemma 1 is exercised for a formation with simultaneous, uncoordinated maneuvers. For the example in this section, the mean orbit element states used are

$$\bar{\text{œ}} = \begin{bmatrix} \bar{a} & \bar{e} & \bar{i} & \bar{\omega} & \bar{\Omega} & \bar{M} \end{bmatrix}^T \tag{17}$$

where $\bar{a}$ is the semi major axis, $\bar{e}$ is the eccentricity, $\bar{i}$ is the inclination, $\bar{\omega}$ is the argument of periapsis, $\bar{\Omega}$ is the ascending node, and $\bar{M}$ is the mean anomaly. The propagated dynamics include Keplerian orbital motion with J_2 perturbations. To emphasize the central results of this paper, no navigation uncertainty or control input noise is considered. The formation consists of three spacecraft. The nominal design barycenter is

$$\bar{\text{œ}}^{b,d} = [R_{eq} + 500\text{ km} \quad 0.05 \quad 28^\circ \quad 30^\circ \quad 45^\circ \quad 150^\circ]^T \tag{18}$$

where R_{eq} is the radius of Earth's equator. The desired formation slots were made arbitrarily by setting δi, $\delta\omega$, $\delta\Omega$, and δM for each of the spacecraft in the formation, and computing δa and δe to make the formation slots J_2 invariant to the designed barycenter in Eq. (18). The formation slots that are used for the simulations are given in Table I. In the simulation each spacecraft is initialized with states that are substantially perturbed from desired formation slots.

TABLE I
DESIRED FORMATION SLOTS

$\delta\bar{\boldsymbol{\mathfrak{E}}}^i_{r,0}$	SC1	SC2	SC3
δa	-2.558 m	-2.558 m	0.0255 m
δe	4.607e-4	4.607e-4	4.628e-6
δi	$0.01°$	$0.01°$	$0.0001°$
$\delta\omega$	$0.1\cos(\bar{\boldsymbol{\mathfrak{E}}}^b(3))°$	$-0.33\cos(\bar{\boldsymbol{\mathfrak{E}}}^b(3))°$	$-1°$
$\delta\Omega$	$-0.1°$	$0.1°$	$0°$
δM	$0°$	$0.2°$	$1°$

To demonstrate the veracity of the sufficient condition (7) derived in Lemma 1, at each control execution time step (arbitrarily chosen to be every 25 seconds), formation spacecraft simultaneously execute maneuvers drawn from a random distribution satisfying the spherical admissible region (Corollary 4) for an uncoordinated leader/follower formation (Corollary 5). In this example with 3 spacecraft, spacecraft 1 is the leader and may not maneuver ($w^1 = 1$ and $w^2 = w^3 = 0$).

Figure 1 plots the formation error Lyapunov function (4) for 16 separate simulated formation responses. No effort is made to minimize maneuver size or fuel usage - it is the purpose of this example to emphasize the sufficient condition inequality.

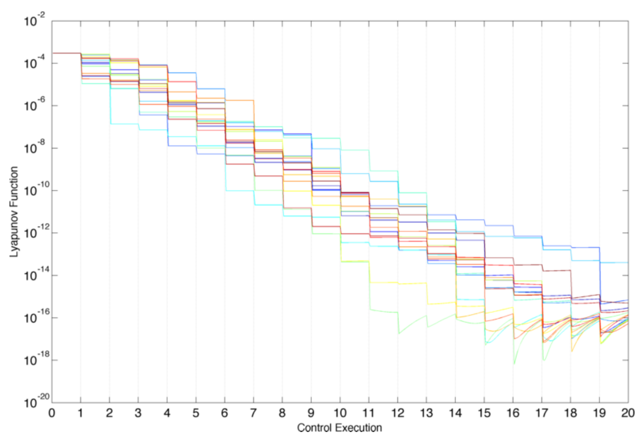

Fig. 1. Formation Lyapunov error for uncoordinated simultaneous leader/follower formation maneuvers. 16 cases are simulated and shown

As seen in Figure 1, the Lyapunov error is consistently driven down using randomly chosen admissible stabilizing controls. As the Lyapunov error becomes sufficiently small, J_2 perturbations induced by small errors with the updated formation orbits and the J_2-invariant orbit design can be seen to limit the Lyapunov error descent to a steady state $V \approx 10^{-16}$. This level of error is consistent with numerical precision error. Operationally, this level of error is orders of magnitude lower than propulsion system impulse and formation state estimation error. Based on the results shown in Figure 1 with random maneuvers in spacecrafts 2 and 3, it is clear that the formation stability sufficient condition (7) is consistent with and verified by simulated results.

V. CONCLUSIONS

The background motivation, literature, and state of distributed formation flight is introduced. Previous related work is reviewed, wherein the concept of a formation barycenter, formation slot, and distributed formation is formally defined defined and discussed. A distributed formation Lyapunov error function is constructed, and combined with the definitions for the barycenter and differential mean orbit formation error, used to rigorously derive sufficient conditions for simultaneous uncoordinated impulsive formation maneuvers. Solution existence of maneuvers that minimize the Lyapunov error is proven. Special cases for single-impulse and simultaneous multiple-impulse maneuvers are discussed, and further simplifications are discussed. A simulation is used to demonstrate the validity of the analytical results in this paper. Potential future work may investigate the case of a formation with arbitrary weights and uncoordinated simultaneous maneuvers or examine regions of admissible maneuvers in $\Delta\mathbf{v}^i$ space to find acceptable and optimal real controllers

VI. ACKNOWLEDGEMENTS

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