DEMPSTER-SHAFER THEORY APPLIED TO ADMISSIBLE REGIONS

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The admissible region approach is often used a bootstrap method to initialize a Bayesian state estimation scheme for too-short-arc measurements. However, there are ambiguities in how prior probabilities are assigned for states in the admissible region. Several approaches have proposed methods to assign prior probabilities, however there are inconsistencies in how the prior probabilities can be manipulated. The application of Dempster-Shafer evidential reasoning theory to the admissible region problem can avoid these ambiguities by eliminating the need to make any assumptions on the prior probabilities. Dempster-Shafer theory also enables the testing of the validity of the assumptions used to construct the admissible region. This paper introduces Dempster-Shafer theory and formulates the admissible region in terms of plausibility and belief which reduce to traditional Bayesian probability once there is sufficient information in the system.

INTRODUCTION

The observation and characterization of space debris is a high priority research area currently. Over 20,000 objects larger than 10 cm are tracked by the Space Surveillance Network and it is estimated that high profile collisions are likely to occur every 5 to 9 years [1] [2]. One of the challenges posed by the problem of characterizing space objects is the initialization of estimators from short observations or measurements. For long observation periods, there are methods available to provide an initial state estimate which may then be used to initialize an Bayesian estimation scheme [3] . However, over sufficiently short observation periods relative to the dynamics of the problem these methods tend to fail due to the lack of observability in the system. In these short arc observations, a continuum of possible states solutions exist and there is much literature on the development of initial state estimation tools for this problem.

The admissible region method offers a way to generate a bounded set of solutions consistent with the observation based on a set of hypothesized constraints [4] [5]. This set can then be used to initialize Bayesian estimation schemes, but first probabilities must be assigned to the states contained within the admissible region. However, the assignment of probability to states in the admissible region is an open area of research. Fujimoto et. al. showed that in the absence of any other information, each state in the admissible region is equally likely to be true and thus a uniform probability distribution should be assigned [6]. Worthy and Holzinger introduce a method to account for systemic uncertainties when forming the admissible region, leading to a fuzzy set for which the probability distribution is equal to the normalized probability of set membership [7]. DeMars and

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Jah utilize a Gaussian Mixture Model (GMM) approach to tractably represent a probability distribution over the admissible region [8]. Hussein et. al. use probabilistic admissible regions generated by mapping a uniformly sampled admissible region from an alternative state space to generate a non-uniform probability distribution [9]. The inconsistency among these treatments of probability in the admissible region is partially addressed in work done by Worthy and Holzinger [10] which shows that the prior probability for the admissible region should be preserved regardless of what state space is selected for the admissible region. However, even this work is based on the application of the Principle of Transformation Groups which is an alternative view of traditional Bayesian statistics [11].

The problem arises because the system is unobservable, which makes a traditional application of Bayesian probability difficult. Bayesian probability requires that there is either support for or evidence against a given hypothesis. In general, when the problem is unobservable no such definitive support is available given a single measurement. While states within the admissible region must support the hypothesized constraint being true, this in itself does not offer support for any one state being the true solution. Treating this region as a PDF is probabilistically incorrect due to the fact that this region is just a diffuse prior (equivalently an uninformative prior) which violates the the principles of probability theory [12]. An alternative branch of probability, Dempster-Shafer theory, deals with this problem by introducing plausibility as a third option which enables assignment of plausibility to states which neither directly support nor refute a hypothesis [13] [14]. The application of Dempster-Shafer theory to the admissible region problem can address the ambiguities that exist in the assignment of prior probability by recasting the admissible region as a region of plausibility. This falls directly in line with the Dempster-Shafer method since while the hypothesized constraints bound the set of potential solutions, they do not directly support any one solution but are all plausible solutions to the problem. A primary theoretical construct of Dempster-Shafer theory is the frame of discernment which contains the propositions which must be assigned belief mass. The proper construction of the frame of discernment for the admissible region problem enables the constraint hypothesis itself to be tested along with the individual states in the admissible region. This provides the ability for a sequential estimator potentially identify when a hypothesized constraint is incorrect and another hypothesis should be used. This work details a generalized form of the frame of discernment for the admissible region problem and uses it to define a Dempster-Shafer sequential estimation scheme. A unique feature of Dempster-Shafer theory is the concept of a probability bound provided by belief and plausibility. While pignistic probabilities can be determined from belief, it can be shown also that as information, or evidence, is gathered belief and plausibility collapse to a single value, the probability. The point when belief and plausibility become equal could, in general, indicate observability in dynamical systems and signify that a traditional Bayesian estimator could be initiated with the now fully defined probability distribution.

The purpose of this paper is to apply Dempster-Shafer theory to the admissible region problem and show how this belief and plausibility based approach evolves into a standard Bayesian approach once enough information is known. This paper will 1) derive a rigorous application of belief and plausibility to the admissible region using Dempster-Shafer theory, 2) define a basic belief assignment function for admissible regions which incorporates systemic uncertainties, 3) define a frame of discernment for the admissible region problem capable of testing the validity of the constraint hypotheses used to construct the admissible region and 4) a discussion of the implementation of belief and plausibility in Bayesian estimation schemes.

THE ADMISSIBLE REGION

The admissible region is introduced to enable the bounding of a continuum of possible states consistent with a given measurement. A given general nonlinear measurement model is defined as follows

$$\mathbf{y} = \mathbf{h}(\mathbf{x}; \mathbf{k}, t) \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state, $\mathbf{k} \in \mathbb{R}^z$ is a set of parameters, and $t \in \mathbb{R}$ is the time. The admissible region introduces a partitioning of the state vector into determined states $\mathbf{x}_d \in \mathbb{R}$ which directly affect the measurements and undetermined states $\mathbf{x}_u \in \mathbb{R}^u$ which do not affect the measurement. Eqn. (1) can then be rewritten as

$$\mathbf{y} = \mathbf{h}(\mathbf{x}_d; \mathbf{k}, t) \tag{2}$$

i.e. there is a one-to-one mapping between $\mathbf{x}_d$ and $\mathbf{y}$. It is clear from Eqn. (2) that any values of $\mathbf{x}_u$ generate the same measurement and thus a continuum of potential solutions exist. The admissible region approach utilizes hypothesized constraints to bound this continuum. The i^{th} hypothesized constraint is defined as

$$\kappa_i(\mathbf{x}_u, \mathbf{y}, \mathbf{k}, t) \leq 0 \tag{3}$$

where the admissible region corresponding to this constraint is then typically defined as

$$Æ_i = \{\mathbf{x}_u \in \mathbb{R}^u \mid \kappa_i(\mathbf{x}_u, \mathbf{y}, \mathbf{k}, t) \leq 0\} \tag{4}$$

Eqn. (4) defines a set in $\mathbb{R}^u$ for which the constraint hypothesis is satisfied. Fujimoto et. al. showed that each member of this set has an equal probability of being the true state, and as such applied a uniform probability density over the admissible region [6].

The problem, as outlined in the introduction, is that there are inherent ambiguities in the assignment of probability in the Bayesian sense. Worthy and Holzinger highlight this issue and proposed that the admissible region must remain remain a region of uniform probability until the problem is observable [15]. The purpose of this paper is to avoid the issue of Bayesian probability assignment altogether by applying Dempster-Shafer theory to the admissible region problem. The next section introduces some of the foundational principles of Dempster-Shafer theory that is then applied to the admissible region problem.

DEMPSTER-SHAFER THEORY

Traditional Bayesian probability is based on the pair (p, q) where p represents the probability that some hypothesis is true and q is the probability that some hypothesis is false [16]. Given some hypothesis h and state x, this is typically formulated as follows

$$p(x|h) \in [0, 1] \tag{5}$$
$$q(x|h) = 1 - p(x|h) \tag{6}$$

The limitation of traditional Bayesian probability is that the state of interest can only support or refute the hypothesis. However, in many real world applications there often exists states which do not inherently refute a hypothesis, but which also do not directly support the hypothesis. The

Dempster Shafer (DS) approach involves utilizing the triple (p, q, r) which adds a quantification of plausibility, r, to address this [17].

To introduce DS theory, first define Ω as the frame of discernment, the set which contains the states for over which evidence, or more specifically belief mass, should be assigned [13]. In traditional DS theory, Ω is defined as

$$\Omega = \{\omega_1, \omega_2, \cdots, \omega_n\} \tag{7}$$

where Ω is a mutually exclusive set of hypotheses for which exactly one hypothesis ω_i is true. The frame of discernment is also called the truth set since it contains the truth solution. DS theory then utilizes mass functions to assign mathematical probability to the power set of Ω which contains the $2^n - 1$ non-empty subsets of Ω (including Ω itself) as well as the empty set. Let $A \subset \Omega$ be a generic subset of the frame of discernment. The mass function is then defined as $m : 2^\Omega \rightarrow [0, 1]$, or equivalently the basic belief assignment (BBA) [18]. A given BBA must satisfy

$$\sum_{A \subseteq \Omega} m(A) = 1 \tag{8}$$

A BBA could, for example, be defined as a plausibility measure on Ω. A simple form of such a plausibility measure is the membership function for $\omega \in A$ [18]. In general, many different choices exist for the BBA and while it is often useful to define a more meaningful BBA based on the application, the selection of the BBA is subjective [19].

The mass function, or BBA, is used to define three useful quantities in DS theory,

$$\forall A \subseteq \Omega, \mathrm{Bel_i(A)} = \sum_{\emptyset \neq B \subseteq A} m_i(B) \tag{9}$$

$$\forall A \subseteq \Omega, \mathrm{Pl_i(A)} = \sum_{A \cap B \neq \emptyset} m_i(B) \tag{10}$$

$$\forall A \subseteq \Omega, \mathrm{Q_i(A)} = \sum_{A \subseteq B} m_i(B) \tag{11}$$

$\mathrm{Bel}_i(A)$ is the belief function and gathers evidence to support proposition A. $\mathrm{Pl}_i(A)$ is the plausibility function gathers evidence which permits the occurrence of proposition A but does not necessarily support A directly. $\mathrm{Q}_i(A)$ is called the commonality function by Shafer and is generally not used in a technical sense [13]. The belief and plausibility are related through duality

$$\forall A, \mathrm{Pl_i(A)} + \mathrm{Bel_i(\bar{A})} = 1 - m_i(\emptyset) \tag{12}$$

where $\bar{A}$ is the complement of A. If $m_i(\emptyset) = 0$ then it is implied that the solution must in Ω. Furthermore, it is only necessary to define one of the quantities, Bel, Pl, or m_i as each of the other quantities may be derived if the BBA is known or if either a belief or plausibility function is directly obtained. This is useful in applications where the BBA may be unknown but a plausibility function or belief function can be directly defined.

Another useful concept introduced from DS theory is the belief-plausibility gap, which is an indicator of ignorance in the system as utilized by Jaunzemis and Holzinger [20].

$$\mathrm{Ig_i(A)} = \mathrm{Pl_i(A)} - \mathrm{Bel_i(A)} \tag{13}$$

Plausibility and belief are shown to be upper and lower bounds on the true probability by Dempster [18]. The belief-plausibility gap can then be used as an indication of the amount of information in the system. It can be used to indicate when there is enough information for an under-determined state estimation problem to have a unique solution.

These summarized components of DS theory will be rigorously applied to the admissible region problem in the next section. This formulation avoids ambiguities in how to address the probabilistic nature of the admissible region for estimation by introducing belief and plausibility rather than just a probability or likelihood.

BELIEF FUNCTIONS ON REAL NUMBERS

Traditional DS theory is derived for scenarios where Ω represents a finite set and the belief functions are defined over the power set of Ω. For general belief functions defined on real numbers, the belief functions are no longer defined over the power set of Ω. Let $\Omega = \mathbb{R}^n$ be a frame of discernment defined over the set of real numbers. Let $\mathcal{B}(\mathbb{R})$ denote the Borel sigma-algebra on the set $\mathbb{R}$ and let $\mathcal{A} = \mathcal{B}^1(\mathbb{R}) \times \cdots \times \mathcal{B}^n(\mathbb{R})$ be the cross product of n such Borel sigma-algebra. The belief density for $\Omega = \mathbb{R}^n$ is then defined as $m : \mathcal{A} \to [0, 1]$ satisfying

$$\int_{\mathcal{A}} m(z) \mathrm{d}z = 1 \tag{14}$$

Note that in general the set $\mathcal{A}$ includes both singleton and non-singleton subsets of $\mathbb{R}^n$. While Eqn. (14) poses no theoretical issues for a frame of discernment defined over real numbers, computationally the problem may become intractable, especially in higher dimensions. It is desired to define a restriction of the frame of discernment to reduce the computational requirements in application.

Consider $\Omega = \mathbb{R}^n$ where the subsets of $\mathcal{A}$ are restricted to the singletons of Ω. The basic belief assignment mass may then be defined as follows

$$m(A) = \begin{cases} m(\mathbf{a}) & A = \{\mathbf{a}\}, \mathbf{a} \in \mathbb{R}^n \\ 0 & otherwise \end{cases}$$

which only assigns belief to the singletons of Ω [12]. Given that Ω is a countable set of points in $\mathbb{R}^n$ then the summations in Eqns. (9) and (10) become infinite sums. Particularly this is useful in state estimation when a particular proposition $A \in \Omega$ cannot take on multiple values. However, Dubois and Prade [18] show that if the subsets of $\mathcal{A}$ are restricted to the singletons of Ω then

$$\mathrm{Bel(A)} = \mathrm{Pl(A)}, \quad \forall \mathrm{A} \in \mathcal{A} \tag{15}$$

This implies that the basic belief assignment $m(\cdot)$ is simply a probability measure on Ω and the desired benefits gained through using DS theory are lost.

Thus, it is desired to have a more inclusive frame of discernment which permits both singleton and nonsingleton propositions while still remaining computationally tractable. Rogers and Costello show that, in general, it is sufficient to consider only a finite number of nonsingleton propositions [21]. The primary nonsingleton proposition which must be included in the frame of discernment is Ω itself, or the uncertainty proposition. The uncertainty proposition enables belief mass to be applied to the entire frame of discernment in addition to the individual singletons. The uncertainty proposition accounts for the fact that there may be situations in which a sensor may not reliably support any individual proposition, but given that the true proposition lies in Ω, it still supports assignment of all belief to Ω. The selection of the other nonsingleton propositions is assumed to be arbitrary, but chosen appropriately for the given problem.

Given that the initial orbit determination problem gives us a frame of discernment over the real numbers, the approach proposed by Rogers and Costello will be applied in this paper. Let the general frame of discernment defined for a belief function on real numbers as

$$\Omega = \{\{x\} \in \mathbb{R}^n\} \cup \mathbb{R}^n \tag{16}$$

which is the union of a countable set of singleton state propositions and the full space $\mathbb{R}^n$. If it is possible to define a region $\mathcal{X}$ inside of which the solution is known to lie such that $\mathcal{X} \subset \mathbb{R}^n$ then the frame of discernment can be written as

$$\Omega = \{\{x\} \in \mathcal{X}\} \cup \mathcal{X} \tag{17}$$

Yet, Eqn. (17) does not account for the fact that evidence can be gathered to suggest the solution does not actually lie in $\mathcal{X}$ as hypothesized. To fully include this possibility that the truth proposition does not lie in $\mathcal{X}$, let $\bar{\mathcal{X}}$ denote the concept of 'none of the above' for estimation. Let the frame of discernment then be defined by

$$\Omega = \{\{x\} \in \mathcal{X}\} \cup \mathcal{X} \cup \mathbb{R}^n \setminus \mathcal{X} \tag{18}$$

This enables belief mass to be assigned outside of the subset in which the solution is thought to lie, giving the ability to identify changes to a system or, for instance, differentiate between objects under observation. Note that the use of $\bar{\mathcal{X}}$ can be generalized into more useful alternative propositions. For instance, if all belief mass is assigned to $\bar{\mathcal{X}}$ given a set of measurements, it could indicate that the original assumptions which were used to construct the admissible region, and thus $\mathcal{X}$, are wrong. Thus, a potentially useful additional proposition is $\tilde{\mathcal{X}}$ which could represent the set of all potential states under a different hypothesis from the one used to create $\mathcal{X}$. This ability to attribute evidence to discriminate between correct or incorrect hypotheses is an example of the utility provided by implementing DS theory for the admissible region problem.

COMBINATION OF EVIDENCE

The belief density defined in the previous section operates on single piece of evidence collected from a given source. An additional utility of DS theory is the flexibility in combining or fusing evidence from different sources. There exists many different forms of rules to combine evidence from different belief assignment functions [22] [23]. A general rule of combination for a given belief function is given by Dempster's combination rule

$$(m_1 \oplus m_2)(A) = \frac{\sum_{B \cap C = A} m_1(B) m_2(C)}{\eta}, \quad A \subseteq \Omega, A \neq \emptyset \tag{19}$$

$$\eta = 1 - \sum_{B \cap C = \emptyset} m_1(B) m_2(C) \tag{20}$$

where the belief functions m_1 and m_2 represent distinct pieces of evidence [18]. For instance, m_1 and m_2 could be a set of two sensors both providing independent evidence about the state of a system. The normalization factor η accounts for the degree of conflict between the two sources.

Dempster's rule is a conjunctive rule that is both commutative and associative and thus can be used iteratively in estimation schemes to update belief assignment.

Dempster's rule is the subject of scrutiny due to potential issues such as Zadeh's paradox which produces yields counterintuitive result if Dempster's rule is applied directly [24]. Dezert et. al. also presents arguments against the use of Dempster's rule of combination in certain situations [25]. In short, the problem arises when there is a source of evidence which get essentially treated as absolute truth, erasing the benefits gained by combining evidence provided by other sources. Problems also arise when the sources of information are not independent of one another as a normalization factor must be included to correct for this dependence [26]. However, application of Dempster's rule to the admissible region problem does not suffer from any of these problems as it is known that any two given measurements are independent of one another. Zadeh's paradox is not an issue due to the treatment of the admissible region problem, since it is unobservable there is no belief mass being directly assigned to particular states, and as such it is unlikely for the evidence gained by a given source to be treated as absolute truth.

THE ADMISSIBLE REGION BBA

The application of DS theory to the admissible region problem is initialized similarly to a traditional Bayesian approach. Since it is of interest to estimate the full state $\mathbf{x}$ which is in $\mathbb{R}^n$ the elements of the frame of discernment Ω must also be in $\mathbb{R}^n$ for this admissible region problem. Following the construction of Ω outlined by Rogers and Costello [21], let $\mathcal{X} \subset \mathbb{R}^n$ represent the full set of admissible states defined by the admissible region and $\bar{\mathcal{X}} = \mathbb{R}^n \setminus \mathcal{X}$ represent the set of all inadmissible states. Mathematically, these sets are represented by

$$\mathcal{X} = \{(\mathbf{x}_u, \mathbf{x}_d) \ : \ \mathbf{x}_u \in \mathcal{AR}\} \tag{21}$$
$$\bar{\mathcal{X}} = \{(\mathbf{x}_u, \mathbf{x}_d) \ : \ \mathbf{x}_u \notin \mathcal{AR}\} \tag{22}$$

and represent the full n dimensional set of admissible states. Then Ω fully is defined as

$$\Omega = \{\mathbf{x} \in \mathcal{X}, \mathcal{X}, \bar{\mathcal{X}}\} \tag{23}$$

which is a fully exhaustive set which must contain the solution. Note also that this is a fully generalizable problem formulation which can be applied to any unobservable system.

A simple outline of the posed problem is as follows, given a measurement $\mathbf{y}$ it is desired to find a BBA of the form $m(\mathbf{x}|\mathbf{y})$ which assigns belief mass to elements of Ω. As noted, the state $\mathbf{x}$ is partitioned into the determined state, which may be directly obtained from the measurements, and the undetermined state which may be unobservable. Assume that at time t_k a measurement $\mathbf{y}_k$ is obtained. Through the independence property applied to belief functions [27], the BBA can be partitioned as

$$m(\mathbf{x}|\mathbf{y}_{0:k-1}) = m_u(\mathbf{x}_u|\mathbf{y}_{0:k-1})m_d(\mathbf{x}_d|\mathbf{y}_{0:k-1}) \tag{24}$$

since knowledge of $\mathbf{x}_d$ does not impact the belief allocated to $\mathbf{x}_u$ and vice versa. The determined states are directly observable through the measurements and as such it is known that the belief mass function is equivalently the probability mass function and Eqn. (25) becomes

$$m(\mathbf{x}|\mathbf{y}_{0:k-1}) = m_u(\mathbf{x}_u|\mathbf{y}_{0:k-1})p_d(\mathbf{x}_d|\mathbf{y}_{0:k-1}) \tag{25}$$

where $p_k(\cdot)$ denotes a probability mass (or density) function. Similarly the plausibility function for the admissible region problem may be defined as

$$\mathrm{Pl}(\mathbf{x}|\mathbf{y}_{0:\mathrm{k}-1}) = \mathrm{Pl}_\mathrm{u}(\mathbf{x}_\mathrm{u}|\mathbf{y}_{0:\mathrm{k}-1})\mathrm{p}_\mathrm{d}(\mathbf{x}_\mathrm{d}|\mathbf{y}_{0:\mathrm{k}-1}) \tag{26}$$

where again the plausibility of the determined state is equal by definition to the belief and thus the probability. It is now of interest to determine the form of either the BBA $m_u(\mathbf{x}_u|\cdot)$ or the plausibility function. Note that the belief assignment and plausibility are both only conditioned on the sequence of measurements $\mathbf{y}_0, \cdots, \mathbf{y}_k$. This contrasts the Bayesian instantiation of an estimation problem where the initial probability is conditioned both on the measurements as well as some a priori distribution. Since DS theory does not require any knowledge of this a priori distribution, it avoids altogether the issue that arises when this a priori distribution is either not known or uninformative.

The BBA for this problem is subject to several constraints from the problem formulation. Given that the states $\mathbf{x}_u$ are undetermined, a single measurement does not offer evidence to substantiate any particular state being more valid than another. In a probabilisitic sense, each state would be given a uniform probability however if there is no evidence for any of the states then belief should not be assigned to any of the states. This would give a vacuous belief function which satisfies the following

$$\mathrm{Bel}(\Omega) = 1 \tag{27}$$

$$\mathrm{Bel}(\mathrm{V}) = 0, \forall V \in \Omega \tag{28}$$

$$\mathrm{Pl}(\mathrm{V}) > 0, \forall V \in \Omega \tag{29}$$

where V is some subset of Ω. A vacuous belief function is equivalently an indication that there is insufficient information to assign belief mass to any given state in the frame of discernment, but since Ω must contain the truth, the whole frame of discernment is attributed all belief mass.

Given the problem formulation, the BBA $m_u(\mathbf{x}_u|\cdot)$ is a vacuous belief function given a single measurement. Through Eqn. (10) it is possible to show that there exists a concise linear relationship between Pl and m given the defined frame of discernment. Let this set of linear equations be represented by

$$\mathrm{Pl}(\mathbf{x}|\mathbf{y}_{0:\mathrm{k}-1}) \propto \mathbf{A}m(\mathbf{x}|\mathbf{y}_{0:k-1}) \tag{30}$$

$$\mathrm{Pl}\begin{pmatrix} \mathbf{x}_1|\mathbf{y}_{0:k-1} \\ \mathbf{x}_2|\mathbf{y}_{0:k-1} \\ \vdots \\ \mathcal{X}|\mathbf{y}_{0:k-1} \\ \bar{\mathcal{X}}|\mathbf{y}_{0:k-1} \end{pmatrix} \propto \begin{bmatrix} 1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} m \begin{pmatrix} \mathbf{x}_1|\mathbf{y}_{0:k-1} \\ \mathbf{x}_2|\mathbf{y}_{0:k-1} \\ \vdots \\ \mathcal{X}|\mathbf{y}_{0:k-1} \\ \bar{\mathcal{X}}|\mathbf{y}_{0:k-1} \end{pmatrix} \tag{31}$$

where the matrix $\mathbf{A} \in \mathbb{R}^{\ell \times \ell}$ where ℓ is the cardinality of Ω. This matrix $\mathbf{A}$ is defined based on the principle of least commitment, the idea behind which implies the BBA defined should never assign more belief mass than justified to elements of Ω [28, 29, 30]. The proportionality accounts for the fact that the inverse relationship may give belief masses that are negative. Since this is nonsensical, proportionality enables the belief function to be corrected accordingly, and since it is the relative masses assigned to each state which maters in the implementation of the filter, the results are equally as valid as if there was an exact relationship directly for $m(\cdot)$. For the remainder of the paper, this relationship will be expressed as an equality without loss of generality.

Since $\mathbf{A}$ is always an invertible matrix, there exists a direct form by which the belief masses can be found assuming the plausibility function is known and is simply given by

$$m(\cdot) = \mathbf{A}^{-1}\text{Pl}(\cdot) \tag{32}$$

Thus it is desired to determine a suitable candidate for the plausibility function, one of which is the probability of set membership in $\mathcal{AR}_i$. Furthermore, since the combination of vacuous belief functions is still vacuous [12], it is desired to find a way to combine plausibility which may be used with Eqn. (32) to find a non-vacuous belief function.

A potential candidate for the plausibility function is the probability of set membership. Let $M_i : \mathbb{R}^u \times \mathbb{R}^{z+1} \to \mathbb{R}^+$ denote the probability of set membership for a given state given the i^{th} hypothesized constraint

$$M_i[(\mathbf{x}_u \in \mathcal{AR}_i)] = \frac{1}{2}\left[1 + \text{erf}\left(\frac{\|\mathbf{x}_\text{u} - \mathbf{x}_{\text{u},\mathcal{B}_\perp}\|}{\sqrt{2\text{tr}\mathbf{P}_{\mathbf{x}_{\text{u},\mathcal{B}_\perp}}}}\right)\right] \tag{33}$$

where the method to determine $\mathbf{P}_{\mathbf{x}_{u,\mathcal{B}_\perp}}$ is given by Worthy and Holzinger [7]. Then let the plausibility function for the admissible region problem be defined by

$$\text{Pl}_\text{u}(\mathbf{x}_\text{u}|\mathbf{y}) = \prod_{i=1}^{c} M_i(\mathbf{x}_u) \tag{34}$$

where c is the total number of constraint hypotheses. The product of the membership functions enable multiple constraint hypotheses to be considered at once.

Given Eqn. (34), a well defined plausibility function for a singleton hypothesis $\mathbf{x} \in \Omega$ may be defined as follows

$$\text{Pl}(\mathbf{x}|\mathbf{y}) = \prod_{i=1}^{c} M_i(\mathbf{x}_u) p(\mathbf{x}_d|\mathbf{y}) \tag{35}$$

But Ω also contains two nonsingleton hypotheses, $\mathcal{X}$ and $\bar{\mathcal{X}}$ which enable belief mass to be allocated to the whole set of admissible states or to the fact that the truth does not lie in the admissible region, respectively. Define

$$\text{Pl}(\mathcal{X}|\mathbf{y}) = \max_{\mathbf{x}\in\mathcal{X}} \text{Pl}(\mathbf{x}_\text{u}|\mathbf{y}) \tag{36}$$

$$\text{Pl}(\bar{\mathcal{X}}|\mathbf{y}) = 1 - \max_{\mathbf{x}\in\mathcal{X}} \text{Pl}(\mathbf{x}|\mathbf{y}) \tag{37}$$

as the plausibility functions for these nonsingleton hypotheses. Eqn. (36) assigns plausibility to the collective set of states comprising the admissible region based on the largest plausibility of an individual state in the admissible region. In general, $\text{Pl}(\mathcal{X}|\mathbf{y}) = 1$ since given a single measurement the admissible region always contains a set of states with probability of set membership equal to 1. The plausibility of the proposition that the true state does not lie in the admissible region is then defined in Eqn. (36). This implies that given a single measurement, since the admissible region hypothesizes that the trues lies in AR, the belief mass assigned to $\bar{\mathcal{X}}$ should be zero until evidence is gained that suggest otherwise.

With Eqns. (35), (36), and (37), a plausibility function is fully defined for Ω, and given the relationship given in Eqn. (32) the belief mass function can also be derived. The remaining concern is the combination of evidence from two independent observations. It is shown that given a single observation the BBA is vacuous and utilizing Eqn. (32) with the plausibility function just defined confirms this fact. The combination of two observations should yield a joint BBA which is no longer vacuous, however without knowledge of the form of the vacuous belief function the combination rule cannot be directly applied. It is desired to find an equivalent combination rule for plausibility, that is given $\mathrm{Pl}_1(\cdot|\mathbf{y}_1)$ and $\mathrm{Pl}_2(\cdot|\mathbf{y}_2)$ what is $\mathrm{Pl}_{1\oplus 2}(\cdot|\mathbf{y}_1, \mathbf{y}_2)$. First consider Dempster's combination rule applied to the admissible region, for any singleton proposition $A \in \Omega$ the combination rule simplifies to

$$(m_1 \oplus m_2)(A) = \frac{m_1(A)m_2(A)}{\eta} \tag{38}$$

since $B \cap C = A$ is also a singleton [31]. Furthermore, if $B \cap C = \emptyset$ then by definition $m_1 = m_2 = 0$ so the normalization term η becomes 1. Thus ,the combination rule as applied to the propositions in admissible region problem is simply given by

$$m_{1\oplus 2}(A) = m_1(A)m_2(A) \tag{39}$$

Applying Eqn. (10), the joint plausibility for the admissible region problem is given by

$$\mathrm{Pl}_{1\oplus 2}(\mathrm{A}|\mathbf{y}_1, \mathbf{y}_2) = \sum_{A \cap B \neq \emptyset} m_{1\oplus 2}(B) \tag{40}$$

which is simply the product of the individual plausibility functions.

$$\mathrm{Pl}_{1\oplus 2}(\mathrm{A}|\mathbf{y}_1, \mathbf{y}_2) = \mathrm{Pl}_1(\mathrm{A}|\mathbf{y}_1)\mathrm{Pl}_2(\mathrm{A}|\mathbf{y}_2) \tag{41}$$

Eqn. (41) now provides an iterative method by which plausibilities from independent measurements can be combined to create a joint plausibility. More importantly, through the use of Eqn. (32), once the joint plausibility is determined, the joint belief mass function can also be determined and thus both belief and plausibility can be found for a given state in Ω.

APPLICATION TO SEQUENTIAL ESTIMATION

The direct application of Eqns. (41) for the admissible region problem is sequential estimation. In particular, the use of the particle filter is the standard estimation tool and it is desired to understand how to incorporate belief and plausibility into the particle filter formulation. There are several existing applications of DS theory to particle filtering which take advantage of either the belief assignment or plausibility functions as the primary weighting terms. Reineking rigorously applies the principles of DS theory to particle filtering deriving a general update equation for plausibility similar to Eqn. (41) [32]. Muños-Salinas et. al. demonstrate the application of DS theory to people tracking by instantiating multiple particle filters with initial belief mass distributions updated with Dempster's rule [26]. The sequential update for plausibility used in this paper is generated through the use of Eqn. (41).

$$\mathrm{Pl}_0(\mathbf{x}|\mathbf{y}_0) = \prod_{i=1}^{c} M_i(\mathbf{x}_u|\mathbf{y}_0)p(\mathbf{x}_d|\mathbf{y}_0) \tag{42}$$

$$\mathrm{Pl}_{\mathrm{k}}(\mathbf{x}|\mathbf{y}_{0:\mathrm{k}}) = \mathrm{Pl}_{\mathrm{k}}(\mathbf{x}|\mathbf{y}_{\mathrm{k}})\mathrm{Pl}_{\mathrm{k}-1}(\mathbf{x}|\mathbf{y}_{0:\mathrm{k}-1}) \tag{43}$$

The initialization of the belief assignment function is vacuous

$$m_0(\mathbf{x}|\mathbf{y}_0) = 0 \tag{44}$$

$$m_0(\mathcal{X}|\mathbf{y}_0) = 1 \tag{45}$$

and the update for the belief assignment function then comes through the inverse linear relationship defined by $\mathbf{A}$

$$m_k(\mathbf{x}|\mathbf{y}_{0:k}) \propto \mathbf{A}^{-1}\mathrm{Pl_k}(\mathbf{x}|\mathbf{y}_{0:\mathrm{k}}) \tag{46}$$

Eqn. (46) is also equal to $\mathrm{Bel}(\mathbf{x}|\mathbf{y}_{0:k})$ by the definition of Ω . The DS particle filter methodology implemented for this work utilizes Eqns. (43) and (46) and is outlined in Algorithm 1.

Algorithm 1 Admissible Region Dempster Shafer Particle Filter

procedure BELIEFFILTERNAME($\mathbf{y}_t$,t,Ω_{t-1},Pl_{t-1})
  $\Omega_t = \emptyset$
  for $k \leftarrow 1, N$ **do**
    $\mathbf{x}^k = \phi(t; \mathbf{x}, t-1), \mathbf{x} \sim \mathrm{Pl_{t-1}}(\Omega_{\mathrm{t-1}}|\cdot)_{\mathrm{t-1}}$ ▷ Sample from plausibility distribution
    $\Omega_t^k = \mathbf{x}^k$
    $\tilde{\mathbf{y}}_t = \mathbf{h}(\mathbf{x}^k, \mathbf{k}, t)$
    $\mathbf{x}_u^k, \mathbf{x}_d^k = \mathbf{g}(\mathbf{x}, \mathbf{k}, \cdot)$
    $p(\mathbf{x}_d^k) = \exp -\frac{1}{2}(\mathbf{y}_t - \tilde{\mathbf{y}}_t)^T\mathbf{R}^{-1}(\mathbf{y}_t - \tilde{\mathbf{y}}_t)$
    $\mathrm{Pl}(\mathbf{x}_u^k) = \mathbb{P}(\mathbf{x}_u^k \in \mathcal{AR}_t)$
    $\mathrm{Pl}(\mathbf{x}^k) = \mathrm{Pl}(\mathbf{x}_u^k)p(\mathbf{x}_d^k)$
  $\mathrm{Pl}_t(\cdot) = \mathrm{Pl}(\cdot)\mathrm{Pl}_{t-1}(\cdot)$ ▷ Update plausibility distribution (Eqn. (43))
  $\mathrm{Pl}_t(\mathcal{X}) = \max_i[\mathrm{Pl}(\mathbf{x}_u^i)]$ ▷ Determine plausibility of $\mathcal{AR}$ (Eqn. (36))
  $\mathrm{Pl}_t(\bar{\mathcal{X}}) = 1 - \mathrm{Pl}(\mathcal{X})$ ▷ Determine 'none-of-the-above' plausibility
  $m_t(\cdot) = \mathbf{A}^{-1}\mathrm{Pl}_t(\cdot)$ ▷ Determine belief assignments (Eqn. (32))
  return m_t, Pl_t, Ω_t

REDUCTION TO BAYESIAN INFERENCE

The use of the concepts of belief and plausibility to instantiate a particle filter for the admissible region problem is a convenient way to avoid the ambiguities inherent in the direct application of Bayesian inference. However, it is still desired to determine when the concepts of belief and plausibility collapse back to standard Bayesian inference. Consider again the linear relationship defined in Eqn. (31). It equivalently states that

$$\mathrm{Pl_k}(\mathbf{x}|\cdot) = m_k(\mathbf{x}|\cdot) + m_k(\mathcal{X}|\cdot) \tag{47}$$

for all the singleton propositions $\mathbf{x} \in \mathcal{X}$. Given that if $\mathrm{Pl}(\mathbf{x}) = m(\mathbf{x})$ then $\mathrm{Pl}(\mathbf{x}) = \mathrm{Bel}(\mathbf{x}) = p(\mathbf{x})$ it is necessary that $m(\mathcal{X}) = 0$ for this construction of a DS particle filter to collapse to the traditional Bayesian implementation. Furthermore, this condition diagonalizes the matrix $\mathbf{A}$ to give a direct correspondance between the belief assignment and the plausibility for not only the states $\mathbf{x}$ but also the 'none-of-the-above' proposition $\bar{\mathcal{X}}$. Since the condition $m(\mathcal{X}) = 0$ yields a probability by construction, it can be related to the observability of the system. Once this condition has been met, it is equivalent to there being enough evidence gathered for the system to be fully observable. Furthermore, once this condition has been met the traditional particle filter algorithm can be used to continue estimation in lieu of Algorithm 1.

SIMULATION AND RESULTS

Three cases will be examined to demonstrate the utility of the application of DS theory to the admissible region problem. The first case demonstrates the use of the DS particle filter as outlined in Algorithm 1 and its reduction to essentially a traditional particle filter once enough observations are made. The second case demonstrates the indifference of DS theory to the problem formulation, showing the benefits gained by choosing to use DS theory to avoid ambiguities caused by different problem formulations in Bayesian theory. The third case demonstrates the utility gained by augmenting Ω with the 'none-of-the-above' proposition, and highlights a potential area for future research.

Scenarios 1 & 2

The observations utilized in the following example test cases are assumed to be captured from Atlanta, GA with an optical telescope with uncertainty parameters given as listed in Table 1. The observation scenario utilizes a measurement function of the form given in Eqn. (2) where

$$\mathbf{y}_k = \begin{bmatrix} \alpha \\ \delta \end{bmatrix} \tag{48}$$

where α is the right ascension and δ is the declination of the space object described in Table 3 relative to the observer. A series of 35 observations are simulated at 20 second intervals. The DS particle filter described in Algorithm 1 is initialized from a uniform distribution and the initial measurement and the belief and plausibility values are updated each time a new simulated observation is acquired. For comparison a traditional particle filter is instantiated with a purely uniform distribution over the admissible region and updated with each new measurement

$$p_{0,PF_1}(\mathbf{x}|\mathbf{y}_0) = \frac{1}{N} \tag{49}$$

where N is the total number of samples. To highlight the independence of DS from this initial distribution, but also to show that the particle filter eventually removes biases caused by arbitrary a priori probability assignment, the following arbitrary 'PDF' is selected for the initialization of the second particle filter

$$p_{0,PF_2}(\mathbf{x}|\mathbf{y}_0) = \frac{1}{C}\sin(\mod(\rho,\pi)) \times \sin(\mod(\dot{\rho},\pi)) \tag{50}$$

where C ensures $p_{0,PF_2}(\cdot)$ sums to 1. The first measurement $\mathbf{y}_0$ is used to construct the admissible region from which each of the samples of $\mathcal{X}$ are drawn. Figure 1 shows the initial distributions for each filter.

Figures are included to convey how the plausibility and belief surfaces evolve over time compared to the probability density function of the particle filter. The purpose of these scenarios is to simply demonstrate that the use of either results in the convergence of the filter to the truth solution and to also demonstrate that in general the particle filter converges to the truth regardless of how the initial distribution is choses. Furthermore, note that due to the independence of the DS particle filter from any prior distribution, it doesn't matter how the a priori distribution is treated as long as the initial distribution represents that of a vacuous belief function. Note also that ultimately plausibility and belief have the same general shape and as states gain additional belief mass they proportionately lose plausibility until belief and plausibility are equal and at this point traditional particle filter implementation is essentially equivalent. Figure 4 shows the near equality of the belief and plausibility values just before $m(\mathcal{X})$ goes to zero at time $t = 600s$.

It is of interest to examine how the belief mass attributed to both $\mathcal{X}$ and $\bar{\mathcal{X}}$ change over the course of the simulation as well. Figure 5 displays these belief masses over the course of the simulation. As can be seen, as more observations are made, and equivalently more evidence is gathered, the evidence supporting assignment to the entire admissible region instead of individual states in AR is reduced to zero. Furthermore, the belief mass assigned to the set $\bar{\mathcal{X}}$ goes toward a large value as well, which is expected since as more evidence is gathered, the belief masses assigned to states in $\mathcal{X}$ mostly go to zero. The exception is where there is significant belief mass assigned to a given state in $\mathcal{X}$, in this case the truth lies close to this state and thus as can be seen the maximum belief assigned to a singleton in $\mathcal{X}$ is larger than the mass assigned to $\bar{\mathcal{X}}$ which implies the original hypothesis is likely correct and a particle filter could now be instantiated from sampling about the now defined probability distribution. Case 3 will demonstrate when this indication is useful to denote a potential incorrect admissible region hypothesis constraint assumption.

Table 1. Measurement Error and Parameter Uncertainty

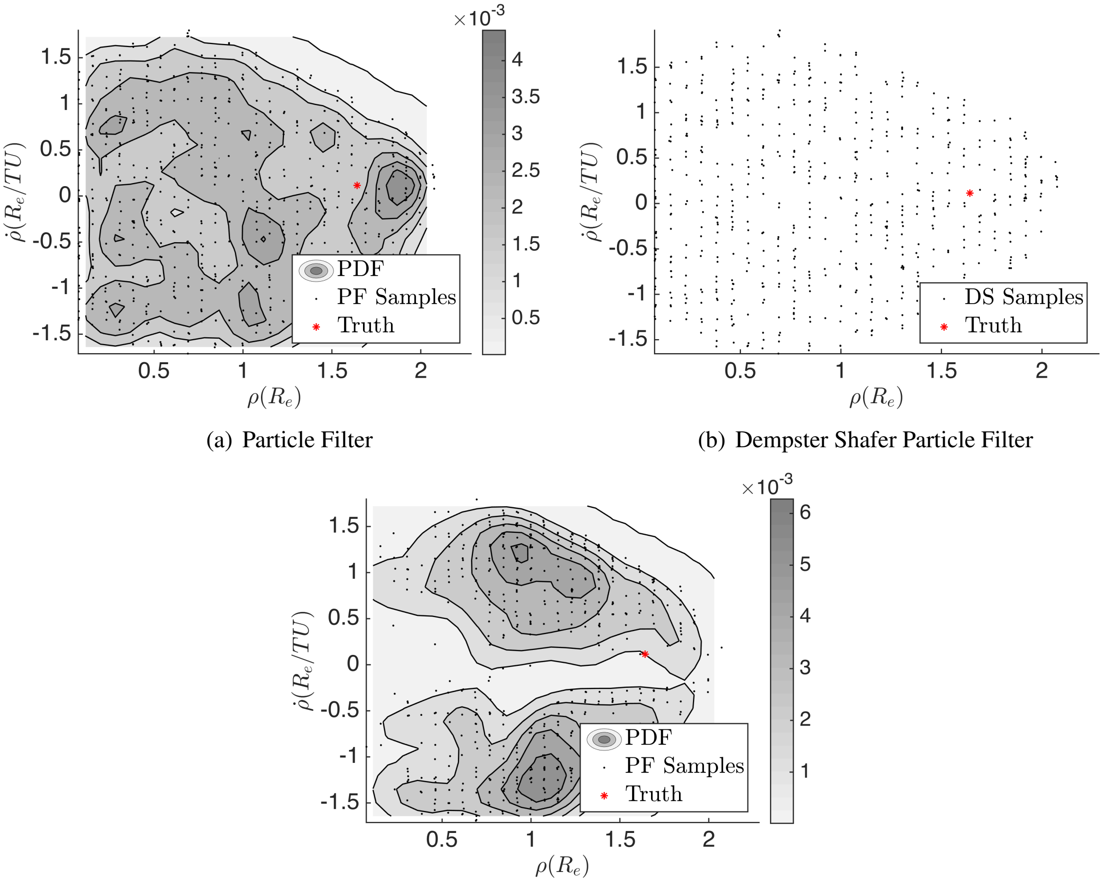
Right Ascension uncertainty, σ_α	50 arcseconds
Declination uncertainty, σ_δ	50 arcseconds
Timing error, σ_t	0.01 s
Position error (each axis), $\sigma_{\mathbf{o}}$	1 m
Velocity error (each axis), $\sigma_{\dot{\mathbf{o}}}$	1 cm/s

Table 2. True Orbit for Case 1 and 2

Semi-major axis, a	6782.0 km
Eccentricity, e	.0007
Inclination, i	51.6°
RAAN, Ω	29.4°
Argument of perigee, ω	117.5°
True Anomaly, ν	20.0°

Scenario 3

Using the same observation configuration as before but now with the object state given in Table 2. Note that this object is in a hyperbolic orbit, which implies that the traditional admissible region constructed for all possible closed orbits does not contain the truth solution. Let AR represent the admissible region under the constraint hypothesis that the object has a closed orbit and let Ω follow from the definition presented in this paper. The set $\mathcal{X}$ represents the set of all admissible closed orbits consistent with the measurements being captured, but it is known that the true state consistent with these measurements lies in $\bar{\mathcal{X}}$. The purpose of this case is to demonstrate that the belief mass assignments can provide indications as to when the assumptions of the problem, namely the assumptions involved in defining AR, and thus $\mathcal{X}$ and also $\bar{\mathcal{X}}$, are valid. In lieu of showing the evolution of the plausibility, belief, and probability surfaces over time, Figure 6 shows the evolution of the belief assignments to the admissible region, the none of the above (NOTA) set, and the maximum belief assigned to any state in the original admissible region similar to Figure 5. The



(a) Particle Filter

(b) Dempster Shafer Particle Filter

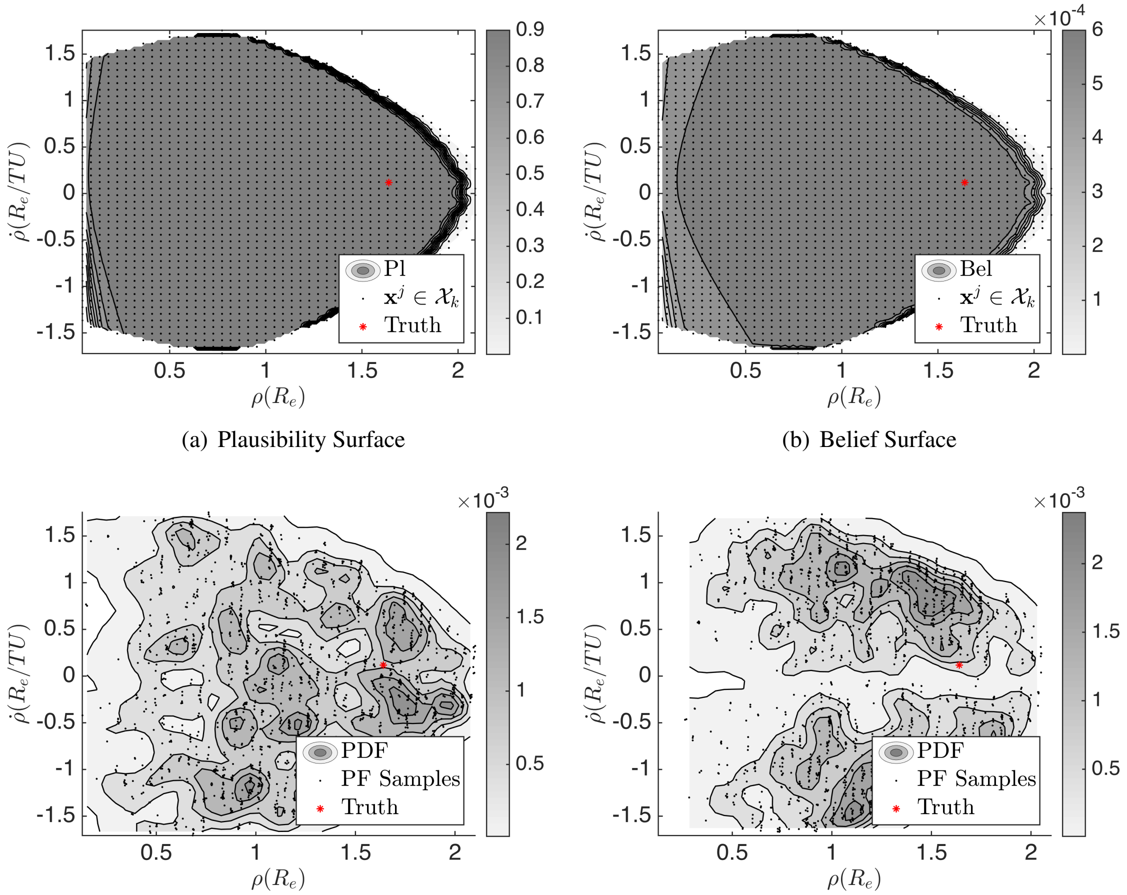
(c) Arbitrary PDF PF

Figure 1. Initial distribution of states for Scenarios 1 and 2.

desire is to show that if more belief mass is being assigned to the 'NOTA' set than any particular state in the admissible region, it may indicate that the truth solution does not lie in the original admissible region. While the 'NOTA' set in this example is simply defined as all states not in the admissible region, it may also be more useful to define subsets of $\mathcal{X} \setminus \mathcal{AR}$ which are valid alternative hypotheses which could be tested such that if belief masses are assigned to these not only is it an indication that the initial hypothesis is incorrect, but also provides an indication as to a correct alternative hypothesis. Figure 6 demostrates that while there is a state in $\mathcal{X}$ which appears to have support, or belief, there is more significant evidence suggesting that the true state is not in $\mathcal{X}$. This information could then be used to reinitialize the DS process with a different, more suitable hypothesis.

CONCLUSIONS

This paper introduces Dempster-Shafer (DS) and applies it to the admissible region problem. Due to the unobservable nature of the problem, there exists ambiguities in how probabilities are assigned to the states within the admissible region. DS theory avoids these ambiguities by utilizing plausibility and belief functions which are derived from a belief assignment which only assigns belief mass if there is direct evidence supporting an given state. Furthermore, it enables the assignment of belief



(a) Plausibility Surface

(b) Belief Surface

(c) PF_1

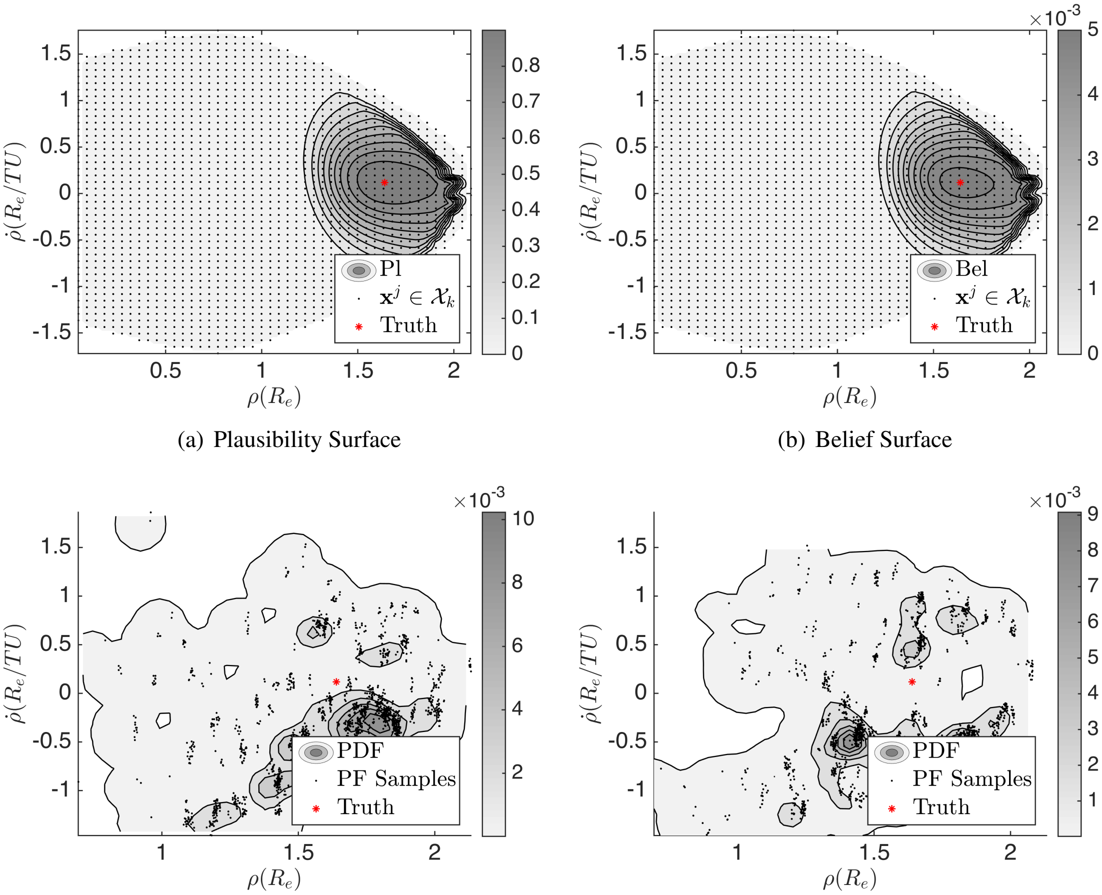
(d) PF_2

Figure 2. Belief, Plausibility, and Probability updates for $t = 20s$.

Table 3. True Orbit for Case 3

Semi-major axis, a	-20000 km
Eccentricity, e	1.5
Inclination, i	51.6°
RAAN, Ω	29.4°
Argument of perigee, ω	117.5°
True Anomaly, ν	20.0°

mass not only to individual states, but also to sets of state solutions, and in particular the entire admissible region. Given a single observation, it is shown that the belief assignment function is vacuous for the admissible region problem and all belief mass is thus assigned to the full admissible region. A plausibility function is defined which assigns plausibility to each state in the admissible region, the admissible region itself, and the proposition that the state does not lie in the admissible region. It is shown that the combination of these plausibility functions enables a corresponding belief function to be defined through a linear relationship which upon sufficient observations collapses to traditional Bayesian inference. This DS particle filter is demonstrated on a few example scenarios



(a) Plausibility Surface (b) Belief Surface

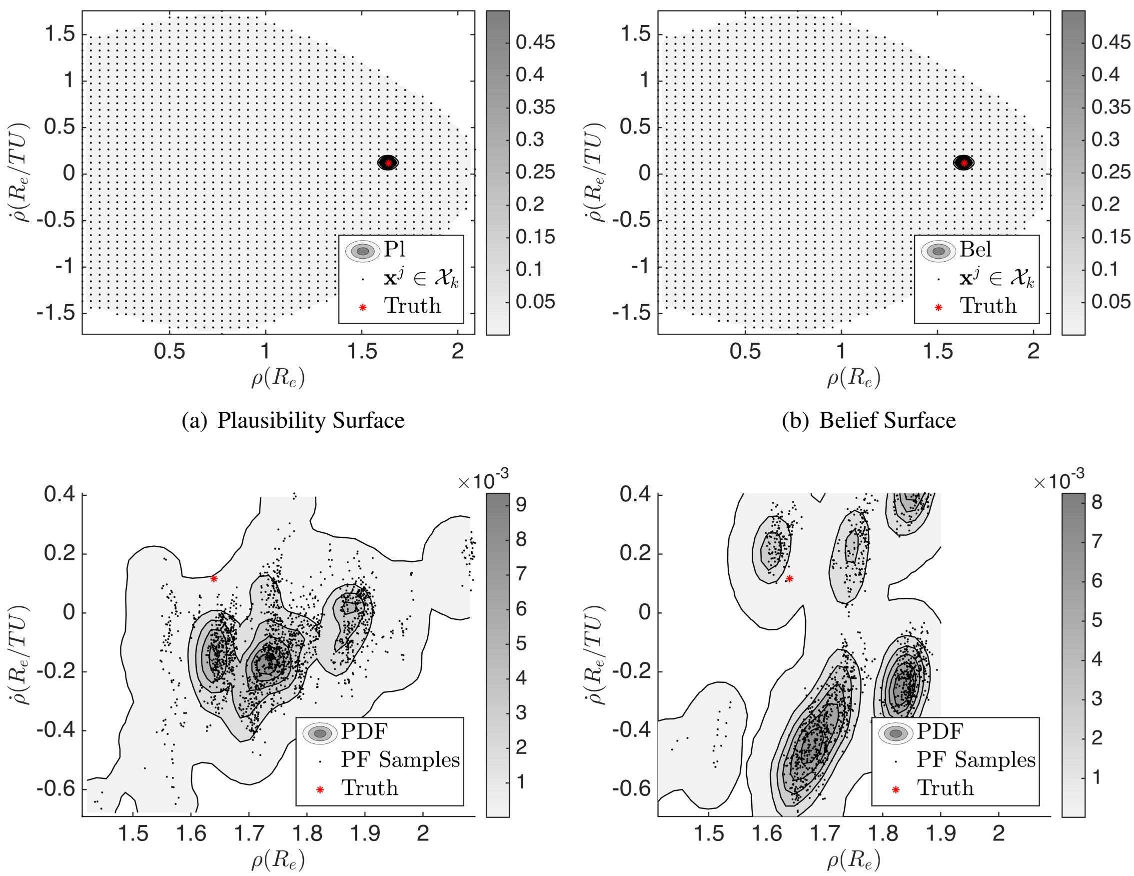
(c) PF_1 (d) PF_2

Figure 3. Belief, Plausibility, and Probability updates for $t = 120s$.

which convey the advantage of utilizing belief functions to initialize sequential estimation schemes for undetermined systems as opposed to application of probability theory.

FUTURE WORK

The application of Dempster-Shafer theory to state estimation offers an innovative way to address potential problems with Bayesian inference. There are numerous additional applications of this theory to the issues faced by the SSA community which may enhance the ability to perform characterization of space objects. An extension of this work could investigate the inclusion of a more exhaustive set of hypotheses which enable a filter instantiated on an incorrect hypothesis to successfully switch to a better hypothesis for tracking. An example of this could be a filter instantiated on a short-arc of an asteroid assumed to be a LEO object. A DS particle filter with an alternative hypothesis for hyperbolic orbits could gather evidence to support switching from a LEO constraint to a hyperbolic one. DS theory can also be applied to the association problem to better understand through plausibility and belief which observations could be, as opposed to, are associated. Furthermore, as is already done in several computer vision fields, DS theory could be applied to the characterization of space objects. Some of these applications will be explored as the topic of



(a) Plausibility Surface (b) Belief Surface

(c) PF_1 (d) PF_2

Figure 4. Belief, Plausibility, and Probability updates for $t = 480s$.

future work in this research area.

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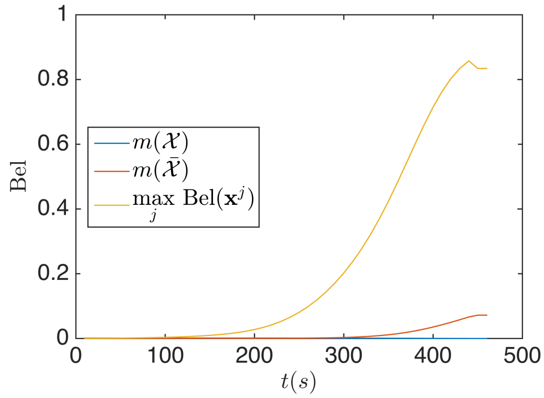

Figure 5. The belief mass assigned to the nonsingletons of Ω

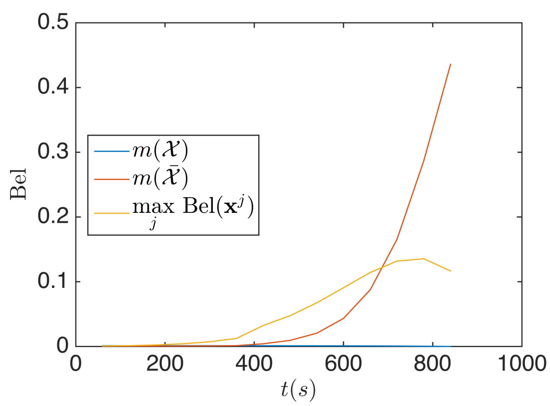

Figure 6. The belief mass assigned to the nonsingletons of Ω indicate the hypothesis may be incorrect.

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