Utilizing Dynamical Systems Concepts in Multidisciplinary Design

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A general multidisciplinary design problem features coupling and feedback between contributing analyses. This feedback may lead to convergence issues requiring significant iteration in order to obtain a feasible design. This work provides a description for casting the multidisciplinary design problem as a dynamical system in order to overcome some of the challenges associated with traditional multidisciplinary design and leverage the benefits of dynamical systems theory in a new domain. Three areas from dynamical system theory are chosen for investigation: stability analysis, optimal control, and estimation theory. Stability analysis is used to investigate the existence of a solution to the design problem. Optimal control techniques allow the requirements associated with the design to be incorporated into the system and allow for constraints that are functions of both the contributing analysis outputs and input values to be handled simultaneously. Finally, estimation methods are employed to obtain an evaluation of the robustness of the multidisciplinary design. These three dynamical system techniques are then combined in a complete methodology for the rapid robust design of a linear multidisciplinary design. The developed robust design methodology allows for uncertainties both within the models as well as the parameters of the multidisciplinary problem. The performance of the developed technique is demonstrated through a linear and nonlinear example problem.

Nomenclature

$(\cdot)_e$	Equilibrium of $(\cdot)$
$(\cdot)^*$	Conjugate transpose of $(\cdot)$, the solution to an equation, or the optimum
$(\cdot)_k$	Iterate k of $(\cdot)$
$(\cdot)_{j\|k}$	Estimate at j given observations up to and including k
$(\cdot)_{nom}$	Nominal value of $(\cdot)$
$\boldsymbol{\beta}$	Deterministic input contribution in the fixed-point iteration equation, $\boldsymbol{\beta} \in \mathbb{R}^{m\times d}$
$\boldsymbol{\delta}$	Bias in the fixed-point iteration equation, $\boldsymbol{\delta} \in \mathbb{R}^{m}$
$\boldsymbol{\gamma}$	Probabilistic input contribution in the fixed-point iteration equation, $\boldsymbol{\gamma} \in \mathbb{R}^{m\times p}$
$\boldsymbol{\Lambda}$	State contribution in the fixed-point iteration equation, $\boldsymbol{\Lambda} \in \mathbb{R}^{m\times m}$
$\boldsymbol{\lambda}$	Lagrange multiplier
$\boldsymbol{\Phi}(k,j)$	Discrete state transition matrix from iterate k to iterate j
ϵ	Convergence tolerance
$\hat{(\cdot)}$	Estimate of the mean of $(\cdot)$
$\lambda_{\max}(\cdot)$	Function which returns the maximum eigenvalue of $(\cdot)$
$\mathbb{R}$	Set of real numbers
$\mathbb{Z}$	Set of all integers
$\mathbb{Z}_+$	Set of all positive integers (*i.e.*, $\mathbb{Z}_+ = \{0, 1, 2, 3, \ldots\}$)
$\mathbf{1}_q$	$q \times 1$ vector of ones
$\mathbf{A_j}$	Matrix describing the state contribution of the j^{th} contributing analysis, $\mathbf{A_j} \in \mathbb{R}^{l_j\times m}$
$\mathbf{B_j}$	Matrix describing the deterministic input contribution of the j^{th} contributing analysis, $\mathbf{B_j} \in \mathbb{R}^{l_j\times d}$

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$\mathbf{C_j}$	Matrix describing the probabilistic input contribution of the j^{th} contributing analysis, $\mathbf{C_j} \in \mathbb{R}^{l_j \times p}$
$\mathbf{d_j}$	Bias associated with the j^{th} contributing analysis, $\mathbf{d_j} \in \mathbb{R}^{l_j}$
$\mathbf{f}(\cdot)$	Concatenation of the contributing analyses input-output relationships
$\mathbf{I_{n\times n}}$	The $n \times n$ identity matrix
$\mathbf{M}$	Matrix describing the linear combination of the pertinent contributing analyses outputs to the design's response, $\mathbf{M} \in \mathbb{R}^{1\times q}$
$\mathbf{N}$	Matrix multiplying the pertinent contributing analyses outputs to the design's response in the Taylor series expansion of the response function, $\mathbf{N} \in \mathbb{R}^{1\times q}$
$\mathbf{Q}$	Covariance of the random noise associated with the model
$\mathbf{R}$	Covariance of the random noise associated with the observation of the system
$\mathbf{u_d}$	Deterministic system-level inputs into the design, $\mathbf{u_d} \in \mathbb{R}^d$
$\mathbf{u_p}$	Probabilistic system-level inputs into the design, $\mathbf{u_d} \in \mathbb{R}^d$
$\mathbf{v}$	Random noise associated with the observation of the system, $\mathbf{v} \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$
$\mathbf{w}$	Random noise associated with the model, $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$
$\mathbf{y_j}$	Contributing analysis output, $\mathbf{y_j} \in \mathbb{R}^{l_j}$
$\mathcal{L}(\cdot)$	Optimal control path cost
$\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$	Normal random variable with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$
$\mathcal{N}(\mu, \sigma^2)$	Normal random variable with mean μ and variance σ^2
$\mathcal{U}$	Set of admissible inputs (controls)
$\mathcal{U}(x_{\min}, x_{\max})$	Uniform random variable which varies between $x_{\min}$ and $x_{\max}$
$\phi(\cdot)$	Terminal state cost
ρ	Density
ρ_{X_i,X_j}	Product-moment (correlation) coefficient between random variables X_i and X_j, $\rho_{X_i,X_j} \in [-1, 1]$
Σ	Set of admissible states
σ_y	Yield strength
$\sigma^2_{X_i}$	Variance of the random variable X_i
$\tilde{(\cdot)}$	Nominal value of $(\cdot)$
$\{\lambda_i\}$	Set of eigenvalues
E	Young's modulus
f	Force magnitude
g	Magnitude of the acceleration due to gravity
I	Mass moment of inertia
l	Length
$L(\cdot)$	Lagrangian
r	Radius
T	Tension
$V(\cdot)$	Lyapunov function candidate
W	Weight
X	Random variable
CA	Contributing analysis
DSM	Design structure matrix

I. Introduction

The design of complex systems is comprised of analyses from numerous disciplines. When each of the disciplines use the same information, have a common set of assumptions, and satisfy the constraints imposed on the design, the design is said to be converged. The convergence process for complex, multidisciplinary designs is typically lengthy and finding an optimal design can be computationally burdensome, particularly for design space exploration when uncertainties are considered. The study of dynamical systems and their associated theory is a well researched field with many established and emerging techniques for their analysis. Exploiting an analogue between the multidisciplinary design problem and dynamical systems enables the leveraging of these resources in a new domain. Casting the multidisciplinary design problem as dynamical system enables leveraging of techniques associated with the dynamical system field in order to overcome some of the traditional shortcomings of multidisciplinary design techniques, such as the computational burden required by the iteration and the potentially conflicting objectives between contributing analysis-level and system-level optimizations.

Finding a converged multidisciplinary design can be thought of as a multidimensional root-finding problem. Due to this, an iteration scheme can be developed for the state vector, where the subsequent iteration relies on information from prior iterates. Furthermore, finding an optimal design is also a root-finding problem. In this work, the process of finding the root iteratively will be shown to be identical to the that of a dynamical system. Therefore, the multidisciplinary design problem can be cast as dynamical system where the state is the iteration-dependent data required by each of the disciplines comprising the design.

The use of concepts from dynamical systems in multidisciplinary analysis and design is not entirely new. Several investigators have applied concepts from dynamical systems in analyzing and designing complex multidisciplinary systems. The use of dynamical systems theory in most of these works has not been explicit and in cases where concepts have been explicitly identified, it has been for a specialized system. For instance, a system where multiple contributing analyses (CAs) collapse to a single CA or there are inherent equations of motion with the application. Some of these previous uses are shown in Table 1.

Table 1. Some previous uses of dynamical system concepts in design.

Investigator	Description	Multidisciplinary	Design Relevant State	Explicit Dynamical System	Stability	Control	Estimation
Appa & Argyris[1]	Simulatenously optimized structure and trajectory of an aircraft; model given by $\mathbf{f}(\mathbf{g}(\mathbf{x}), \mathbf{p})$ and limits coupling between CAs	✓	✓	✓	-	✓	-
Smith & Eppinger[2]	Decomposes an organizational DSM based on the eigenstructure and “modes” of the organization	✓	-	-	✓	-	-
Lewis & Mistree[3]	Uses game theory to decompose an organizational DSM	✓	-	-	✓	-	-
Delaurentis[4]	Uses metamodels to design an aircraft considering stability and trajectory constraints	✓	-	-*	✓	✓	-
Grant[5,6]	Simultaneously performs trajectory and shape optimization; collapses multidisciplinary design into a single analysis	✓†	✓	✓	-	✓	-

*Work by Delaurentis deals with algebraic results from equations of motion

†Work by Grant is a specific instance of multidisciplinary design with no feedback

As opposed to the previous applications in Table 1, this investigation provides the theoretical foundations for casting the general multidisciplinary design problem as a dynamical system, including handling of equality and inequality constraints within the design. Three particular techniques from different domains of dynamical system theory are examined in depth as they directly relate to the development of a rapid robust design methodology. These techniques are:

1. **Stability analysis**: The existence of a converged design (for a given iteration scheme) can be determined in the same way as analyzing a dynamical system's stability, where the conditions for asymptotic stability are identically equal to those required for convergence.

2. **Optimal control**: Equality and inequality constraints on the design variables and outputs of the contributing analyses not explicitly handled by feedback within the design can be handled similarly to state equality and inequality constraints in optimal control theory by adjoining "tangency conditions" to the objective function.

3. **Estimation theory**: A design's robustness characteristics (*i.e.*, the mean and variance) can be analyzed using a Kalman filter (for linear designs), where the mean state and covariance matrix are products from propagating the filter until the design converges. This technique allows for accounting for uncertainties within the model itself as well as within the parameters of the design.

Utilizing all of these techniques as an ensemble allows for a rapid methodology for robust multidisciplinary design to emerge. This technique finds a conservative upper bound of the variance of the design to a scalar objective function.

II. Theoretical Foundations

II.A. The Concept of a State

The concept of a state is fundamental in transforming the multidisciplinary design problem to a dynamical system. It is a summary of the status of the system at a particular instance.[7]

Definition: State

The *state* consists of the minimum set of parameters that completely summarize the internal status of the dynamical system in the following sense: at any time $t_0 \in \mathcal{T}$ the state $\mathbf{x}(t_0)$ is known, then the output at a future instance in time, $t_1 \in \mathcal{T}$, $\mathbf{y}(t_1)$ where $t_1 > t_0$ can be uniquely determined provided the input $\mathbf{u}_{[t_0,t_1]} \in \mathcal{U}$ is known.

In the work that follows, the state will be defined as the output of each of the CAs in the multidisciplinary design.

II.B. Dynamical Systems

A dynamical system uses a fixed rule to describe the evolution of a state. There are two components of a dynamical system, a state vector which provides the state of the system and a function which is the fixed rule describing how the state will evolve.

Definition: Dynamical System

Dynamical systems are functional relationships where a fixed rule describes how a state evolves. It requires:

1. A state variable (or vector) which characterizes the system
2. A fixed rule describing how the state changes

The framework developed for this work relies on discrete dynamical systems. That is, a dynamical system of the form

$$\left.\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k, k) \\ \mathbf{y}_k &= \mathbf{g}(\mathbf{x}_k, \mathbf{u}_k, k) \end{aligned}\right\} \tag{1}$$

where $\mathbf{x}$ is the state of the system, $\mathbf{f}$ is a function which describes the time evolution of the system, $\mathbf{u}$ is the input into the system, and k is the iterate number. A specific instance of Eq. (1) that is used throughout this work is a linear, discrete dynamical system, which is given by

$$\left.\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}_k\mathbf{x}_k + \mathbf{B}_k\mathbf{u}_k \\ \mathbf{y}_k &= \mathbf{C}_k\mathbf{x}_k + \mathbf{D}_k\mathbf{u}_k \end{aligned}\right\} \tag{2}$$

II.C. Multidisciplinary Design as a Dynamical System

II.C.1. Identification of Converged Designs

Identifying converged designs in multidisciplinary systems can be thought of as the process of finding the root of a function. Consider a multidisciplinary problem where the analysis variables are described by a multivariable function $\mathbf{f}(\mathbf{x}, \mathbf{p})$ where $\mathbf{x}$ are the design variables and $\mathbf{p}$ are parameters of the problem. Assume that the requirements of the design are given by only equality constraints that are a function of the performance of the system. The performance of the design is described by a multi-variable mapping $\mathbf{g}(\mathbf{f}(\mathbf{x}, \mathbf{p}))$ and the requirements are given by $\mathbf{z}$. In order to meet the requirements it is necessary to adjust the design variables $\mathbf{x}$ so that

$$\mathbf{z} = \mathbf{g}(\mathbf{f}(\mathbf{x}, \mathbf{p})) \tag{3}$$

Equation (3) can be rewritten as

$$\mathbf{z} - \mathbf{g}(\mathbf{f}(\mathbf{x}, \mathbf{p})) = \mathbf{0} \tag{4}$$

The solution $\mathbf{x}^*$ of Eq. (4) is the root of the system and the process is referred to as root-finding. Since identifying feasible designs within the multidisciplinary design problem requires finding the value of $\mathbf{x}$ that satisfies Eq. (4), this process can be thought of as a root-finding problem when an iterative solution method is chosen.

Many numerical methods for finding the root of a function, $\mathbf{g}(\mathbf{x})$, are dynamical systems since they rely on iterative schemes to identify the root.[8] For instance, the bisection method, secant method, function iteration method, and Newton's method are all iterative techniques that satisfy the requirements of a dynamical system.

To demonstrate, consider Newton's method of finding a root to the unidimensional equation $g(x) = 0$ as shown shown in Fig. 1.

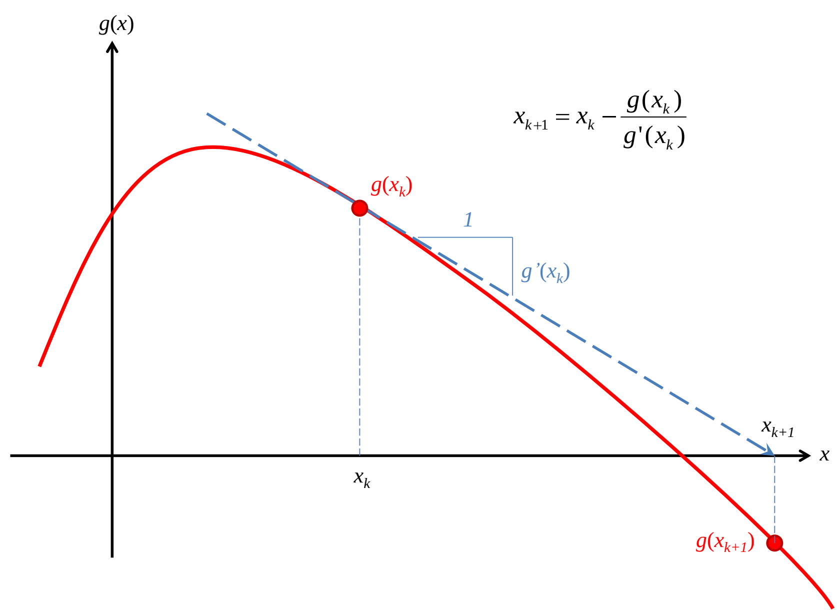


Figure 1. Newton's method for numerically finding the root of a nonlinear equation.

An initial guess is first taken, x_0. Then $y_0 = g(x_0)$ is computed. If $y_0 = 0$, then x_0 is a root. This, however, is usually not the case. Newton's method approximates the slope of the function at a given point in order to find the root. It is desired to find y_1 such that $y_1 = 0$. At x_0, an approximation for the slope is given by

$$g'(x_0) = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{0 - g(x_0)}{x_1 - x_0}$$

When this relationship is rearranged for x_1 the following results

$$x_1 = x_0 - \frac{g(x_0)}{g'(x_0)}$$

This can be generalized for any iterate k

$$x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)}$$

This relationship has the necessary components to by a dynamical system: (1) a state, in this case x, and (2) a fixed rule describing how x evolves with iteration.

II.C.2. Design Optimization

In order for a converged design to be an optimum with respect to some objective function, its performance needs to be evaluated with respect to other potential designs.

The first-order, necessary condition associated with optimization problem given by

$$\left.\begin{array}{ll} \textbf{Minimize}: & \mathcal{J}(\mathbf{x},\mathbf{p}) \\ \textbf{Subject to}: & \mathbf{g}_i(\mathbf{x},\mathbf{p}) \leq \mathbf{0}, \quad i = 1,\ldots,n_g \\ & \mathbf{h}_j(\mathbf{x},\mathbf{p}) = \mathbf{0}, \quad j = 1,\ldots,n_h \\ \textbf{By varying}: & \mathbf{x} \end{array}\right\} \tag{5}$$

require the a stationary point of the Lagrangian to be defined. For the optimization problem given in Eq. (5), the Lagrangian is

$$L(\mathbf{x},\mathbf{p},\boldsymbol{\lambda}) = \mathcal{J}(\mathbf{x},\mathbf{p}) + \sum_{i=1}^{n_g} \lambda_i \mathbf{g}_i(\mathbf{x},\mathbf{p}) + \sum_{j=1}^{n_h} \lambda_{n_g+j} \mathbf{h}_j(\mathbf{x},\mathbf{p}) \tag{6}$$

The first-order, necessary conditions for $\mathbf{x}^*$ to be an optimum are[9]

1. $\mathbf{x}^*$ is feasible
2. $\lambda_i \mathbf{g}_i(\mathbf{x}^*,\mathbf{p}) = \mathbf{0} \quad i = 1,\ldots,n_g$ and $\lambda_i \geq 0$
3. $\nabla_{\mathbf{x}} L(\mathbf{x},\mathbf{p},\lambda) = \nabla_{\mathbf{x}} \mathcal{J}(\mathbf{x},\mathbf{p}) + \sum_{i=1}^{n_g} \lambda_i \nabla_{\mathbf{x}} \mathbf{g}_i(\mathbf{x},\mathbf{p}) + \sum_{j=1}^{n_h} \lambda_{n_g+j} \nabla_{\mathbf{x}} \mathbf{h}_j(\mathbf{x},\mathbf{p}) = \mathbf{0}$ with all $\lambda_i \geq 0$ and λ_{n_g+j} unrestricted in sign

It should be clear that each of the necessary conditions is a root-finding problem by itself. As previously discussed, the process of finding $\mathbf{x}^*$ is a process of root-finding. Additionally, as apparent from the form of the relationships, the other two necessary conditions for optimality may also be obtained through a root-finding technique.

II.C.3. Identifying an Optimal Multidisciplinary Design

Multidisciplinary design optimization can be broken down into two steps: (1) identifying feasible designs and (2) identifying the optimal design from the set of feasible candidates. As discussed, both of these steps are root-finding problems. With the choice of an appropriate iterative numerical root-finding scheme, each of these individual steps can be posed as dynamical systems. When combined together, a nested root-finding problem results, whereby the function being optimized is actually a root-finding problem itself. This is shown in Fig. 2

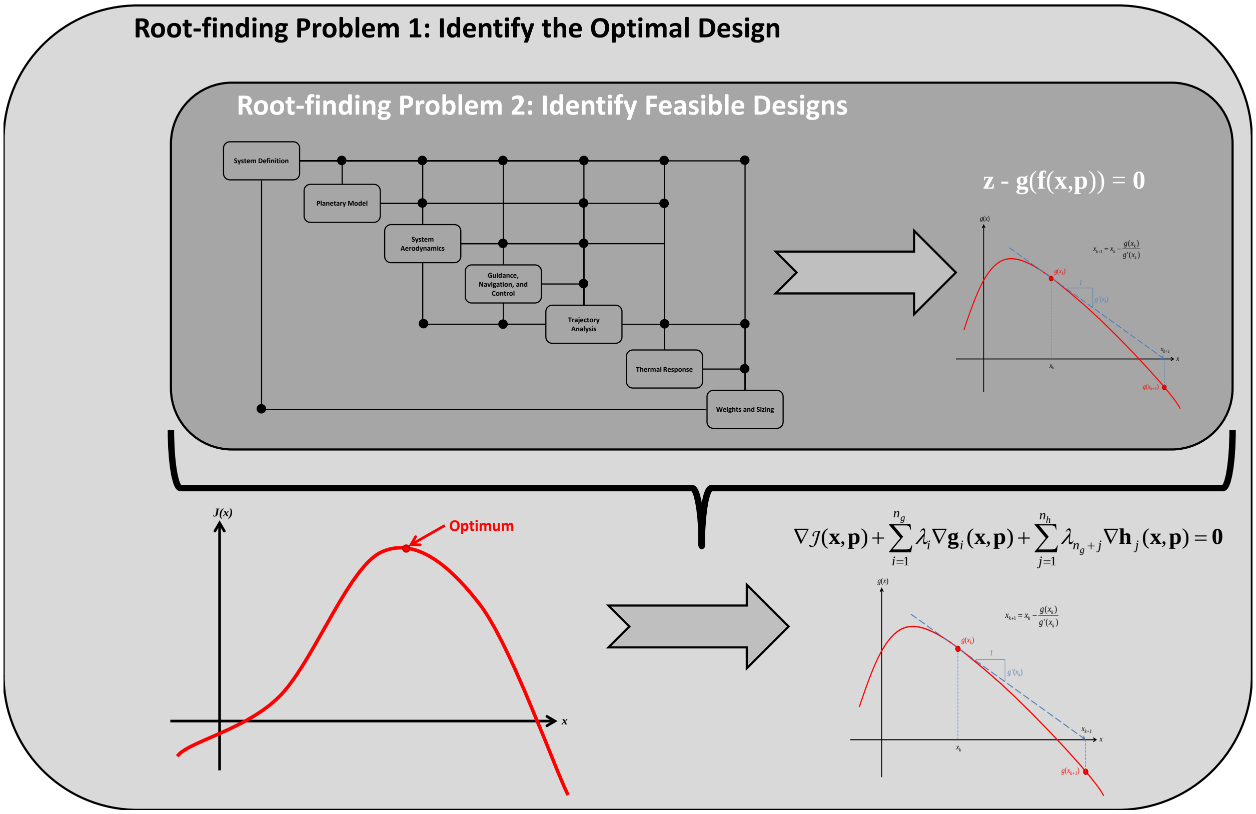


Figure 2. Multidisciplinary design through root-finding.

II.D. Dynamical System Stability Analysis

The concept of stability allows for the identification of feasible designs. For a given initial state, a system is stable if the state does not grow beyond the initial state's magnitude. More rigorously, this is defined in terms of equilibrium points of a system. Consider the discrete dynamical system defined by Eq. (1), the equilibrium point is defined as

Definition: Equilibrium of a Dynamical System

A particular point $\mathbf{x}_e$ is an *equilibrium point* of the dynamical system given by Eq. (1) if the system's state at iterate $k = 0$ is $\mathbf{x}_e$ and $\forall k \in \mathbb{Z}_+ \setminus \{0\}$, $\mathbf{f}(\mathbf{x}_e, \mathbf{0}, k) = \mathbf{x}_e$.

For a linear dynamical system, given by Eq. (2), the equilibrium point is only the origin of the system (*i.e.*, $\mathbf{x}_e = \mathbf{0}$).

The equilibrium point's stability is defined with regard to the zero-input discrete dynamical system given by[7,10–12]

$$\left.\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{f}(\mathbf{x}_k, \mathbf{0}, k) \\ \mathbf{x}_{k=0} &= \mathbf{x}_0 \end{aligned}\right\} \tag{7}$$

Figures 3 and 4 demonstrate the concept of equilibrium point stability. Figure 3 demonstrates different state trajectories for a continuous dynamical system while Fig. 4 shows a more intuitive concept of stability.

Definition: Stability

For the system given by Eq. (7), if $\forall \epsilon > 0$, $\exists \delta(\epsilon, 0) \in (0, \epsilon]$ an equilibrium point of the system is

- *stable* if $\forall k > 0$ and $\| \mathbf{x}_0 \| < \delta$, $\| \mathbf{x}_k \| < \epsilon$
- *asymptotically stable* if
  1. the equilibrium point is stable and
  2. $\exists\ \delta' \in (0, \epsilon]$ such that whenever $\| \mathbf{x}_0 \| < \delta'$ the state's evolution satisfies $\lim\limits_{k \to \infty} \| \mathbf{x}_k \| = 0$
- *unstable* if it is not stable or asymptotically stable

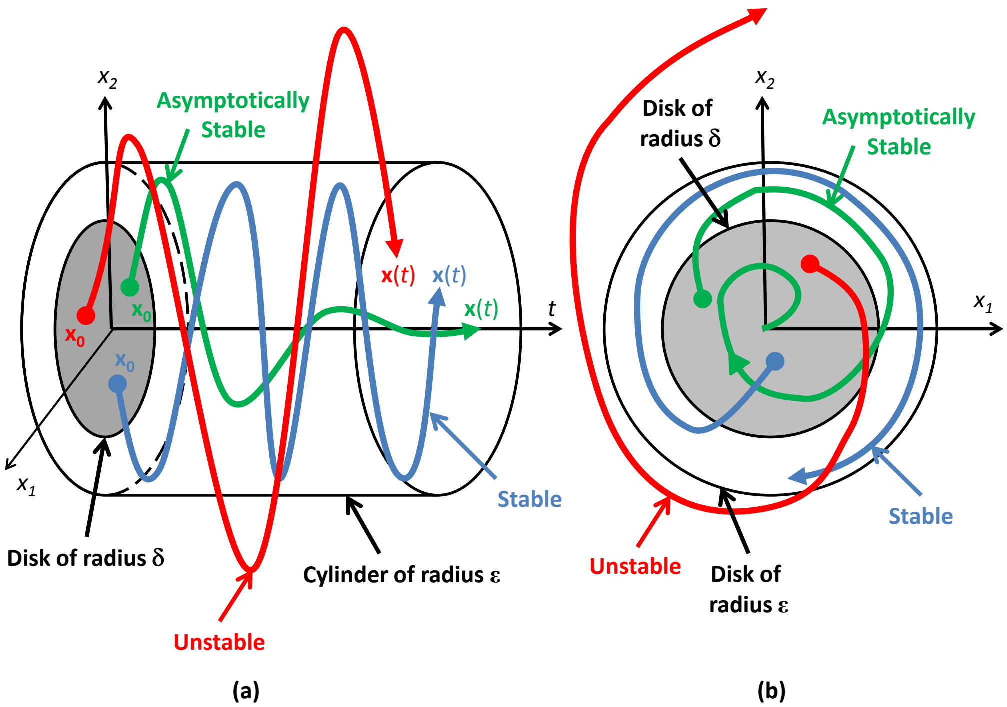


Figure 3. Visualization of state trajectories in (a) $\mathbb{R}^2 \times \mathcal{T}$ and (b) $\mathbb{R}^2$ showing stability for a continuous dynamical system.

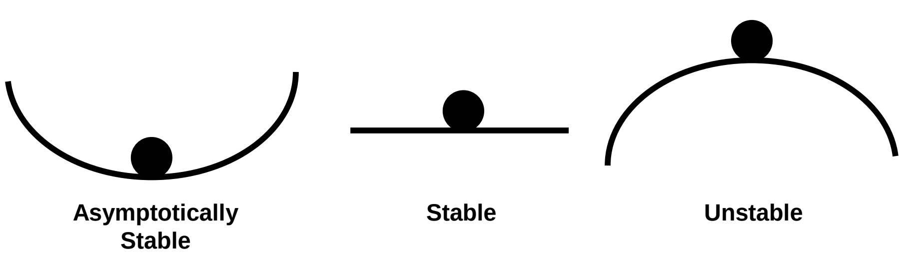


Figure 4. Visualization of the concept of stability.

For discrete, linear systems, that is dynamical systems given by Eq. (2), the solution for the evolution of the state and the output is given by

$$\mathbf{x}_k = \mathbf{\Phi}(k,0)\mathbf{x}_0 + \sum_{j=1}^{k} \mathbf{\Phi}(k,j)\mathbf{B}_{j-1}\mathbf{u}_{j-1} \tag{8}$$

where $\mathbf{\Phi}(k,j)$ is the discrete state transition matrix. This transition matrix is given by

$$\mathbf{\Phi}(k,j) = \mathbf{A}^{k-j} \tag{9}$$

in the case where $\mathbf{A}_k = \mathbf{A}\ \forall k \in \mathbb{Z}_+$, that is when $\mathbf{A}$ is constant. Substituting Eq. (8) and Eq. (9) into Eq. (2) yields[7]

$$\left.\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}^{k+1}\mathbf{x}_0 + \sum_{j=1}^{k} \mathbf{A}^{k-j+1}\mathbf{B}_{j-1}\mathbf{u}_{j-1} + \mathbf{B}_k\mathbf{u}_k \\ \mathbf{y}_k &= \mathbf{C}_k\left[\mathbf{A}^k\mathbf{x}_0 + \sum_{j=1}^{k} \mathbf{A}^{k-j}\mathbf{B}_{j-1}\mathbf{u}_{j-1}\right] + \mathbf{D}_k\mathbf{u}_k \end{aligned}\right\} \tag{10}$$

which is a relationship that depends on the initial condition and the control history. In the unforced case (*i.e.*, $\mathbf{u}_k = \mathbf{0}\ \forall k \in \mathbb{Z}_+$) and by the Cayley-Hamilton theorem, the stability criterion is given in Table 2, where $\{\lambda_i\}$ are the eigenvalues of $\mathbf{A}$.[7,13,14]

Table 2. Linear, constant discrete dynamical system stability criterion.

Classification	Criterion
Unstable	If $\lvert\lambda_i\rvert > 1$ for any simple root or $\lvert\lambda_i\rvert \geq 1$ for any repeated root
Stable	If $\lvert\lambda_i\rvert \leq 1$ for any simple root and $\lvert\lambda_i\rvert < 1$ for all repeated roots
Globally Asymptotically Stable	$\lvert\lambda_i\rvert < 1$ for all roots

For a more general discrete dynamical system (*e.g.*, a dynamical system that does not have constant coefficient or a nonlinear dynamical system), the direct method of Lyapunov can be used to investigate the stability of the system. In this case, a scalar function $V(\mathbf{x})$, also known as a Lyapunov function candidate, which is continuous and has continuous partial derivatives in a region about the origin (equilibrium) gives insight into the system's stability. Table 3 shows the stability criterion using Lyapunov's direct method for discrete dynamical systems where $\Delta V = V(\mathbf{x}_{k+1}) - V(\mathbf{x}_k)$.[7,10–14]

Table 3. Discrete dynamical system stability using Lyapunov's direct method.

Classification	Criterion
Stable	1. $V(\mathbf{x}) > 0$ 2. $\Delta V \leq 0$
Asymptotically Stable	1. $V(\mathbf{x}) > 0$ 2. $\Delta V < 0$
Globally Asymptotically Stable	1. $V(\mathbf{x}) > 0\ \forall \mathbf{x} \neq \mathbf{0}$ and $V(\mathbf{0}) = 0$ 2. $\Delta V < 0\ \forall \mathbf{x} \neq \mathbf{0}$ (or $\Delta V \leq 0\ \forall \mathbf{x}$ and $\Delta V \neq 0$ for any solution sequence $\{\mathbf{x}_k\}$) 3. $V(\mathbf{x}) \to \infty$ as $\parallel \mathbf{x} \parallel \to \infty$

Note that not finding a function that satisfies the conditions in Table 3 for a Lyapunov function does not imply anything regarding the stability of the system.

From the multidisciplinary design perspective, stability of the dynamical system gives information into the convergence characteristics of the design. Asymptotic stability implies that there is a limited region for which the design will converge whereas global asymptotic stability implies that the design will converge with enough iteration regardless of the design assumptions used to start the convergence procedure. If the dynamical system representing the multidisciplinary design is found to be unstable or stable it implies that the design will not converge for that choice of root-finding schemes.

II.E. Including State Constraints through Optimal Control Theory

Consider another dynamical system concept, that of optimal control. Optimal control techniques can be used to adjoin a "tangency" condition for constraints that are a function of the input or state. To demonstrate this technique, instead of the discrete dynamical system problem, consider the general continuous-time optimal control problem given by

$$\left.\begin{array}{ll}\textbf{Minimize}: & \mathcal{J} = \phi(\mathbf{x}(t_f), t_f) + \displaystyle\int_{t_0}^{t_f} \mathcal{L}(\mathbf{x}(t), \mathbf{u}(t), t)dt \\ \textbf{Subject to}: & \dfrac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \\ & \mathbf{u}(t) \in \mathcal{U} \\ & \mathbf{x}(t) \in \Sigma \\ \textbf{By varying}: & \mathbf{u}(t) \end{array}\right\} \tag{11}$$

In Eq. (11), ϕ is the terminal state cost, $\mathcal{L}$ is the transient or path cost, $\mathcal{U}$ is the set of admissible controls, and Σ is the set of admissible states. Suppose that there is a constraint on the state given by

$$\mathbf{S}(\mathbf{x}, t) = 0 \tag{12}$$

Differentiating Eq. (12) with respect to time, one obtains

$$\dot{\mathbf{S}} = \frac{\partial \mathbf{S}}{\partial t} + \frac{\partial \mathbf{S}}{\partial \mathbf{x}} \frac{d\mathbf{x}}{dt} = \mathbf{0} \tag{13}$$

Substituting the state equation into this result yields

$$\dot{\mathbf{S}} = \frac{\partial \mathbf{S}}{\partial t} + \frac{\partial \mathbf{S}}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) = \mathbf{0} \tag{14}$$

and allows for a technique to yield the optimal control, $\mathbf{u}(t)$, that minimizes $\mathcal{J}$ and meets the equality constraint on the state.[15,16] If the control is not explicit in Eq. (13), then the process of differentiating $\mathbf{S}$ and substituting the state equation is continued until the control is explicit in the equation to form a set of q point relationships $\{\mathbf{S}^{(n)}\}, \quad n = 0, \ldots, q-1$, where n is the order of the derivative. These tangency conditions can be adjoined using Lagrange multipliers to the path cost, $\mathcal{L}$, to solve for the optimal control history.

Inequality constraints of the form

$$\bar{\mathbf{S}}(\mathbf{x}, t) \leq \mathbf{0} \tag{15}$$

can be handled similarly.[15,16] In this case, the solution process depends on whether or not the state is on the boundary. If it is on the boundary, the same solution process to equality constraints is followed, while for off-boundary solutions, the terms are ignored. This results in a multiple sub-arc solution, although fundamentally the process is identical to the equality constraint case.

II.F. Propagating Uncertainty

There are many methods for propagating the uncertainty through a system. These are classified in Table 4. In this work, the Kalman filter, similar to that used in linear covariance analysis, is implemented. The details are found subsequent to this survey.

Table 4. **Various uncertainty propagation techniques.**

Method	Description	Example(s)	References
Analytical	Exact propagation in functional form; only applicable for a small subset of problems for which the PDE governing the propagation yields an analytic solution	Liouville and Fokker-Plank-Komogorov Equations	17–20
Sampling	Estimate the uncertainty by running successive deterministic simulations with values chosen from random distributions for the stochastic variables associated with the problem; can be computationally burdensome; metamodeling techniques can be used to reduce burden	Monte Carlo, Response Surface Methodology, and Unscented Transform	21–26
Most Probable Point	Estimate the CDF for probabilistic system design; values chosen from random distributions and evaluate input distribution against a constraint function that is a requirement of; the design generally transform the input distribution into the standard normal space and vary constraint value	Fast Propability Integration	27–38
Covariance	Uses ideas from Kalman filter theory to propagate a normal distribution through a dynamic system described by a differential equation	Linear Covariance	39–41
Other	Approximate the uncertainty distribution using a variety of techniques including bounding analyses and approximations that attempt to solve the PDE describing the propagation of the uncertaintainty	Differential Analysis, Fourier Analysis, Polynomial Chaos, Fast Probability Gaussian Closure, Gaussian Mixture, Stochastic Averaging	42–55

II.F.1. The Kalman Filter

Feedback within the multidisciplinary design problem leads to significantly longer analysis times. Several methods have been developed for use in the design process to eliminate the feedback within the design. The traditional approach to eliminate the feedback within the design-analysis cycle is to enforce a constraint in the *converged* design that the estimated value of the feedback variable is within a given tolerance of the value resulting from the subsequent CA. This is an effective technique for deterministic analysis and design; however, it can be computationally time consuming for robustness assessment and robust design. A novel technique which applies concepts from estimation theory to this challenge is the use of the Kalman filter. This approach is particularly applicable to the robustness analysis problem as the final quantities being sought are the mean and the variance of an objective function. This approach has not been implemented previously because the Kalman filter is typically implemented with respect to a dynamical system and the multidisciplinary analysis and design problem is traditionally concerned with algebraic quantities. However, as discussed previously, by viewing the root-finding problem as a dynamical system, the Kalman filter becomes tractable.

For instance, fixed-point iteration is defined by the relation

$$\mathbf{y}_k = \mathbf{f}(\mathbf{y}_{k-1}), \quad \forall k \in \mathbb{Z}_+ \setminus \{0\} \tag{16}$$

where $\mathbf{f}(\mathbf{y}_{k-1})$ is the output value of the CAs on the $k^{\text{th}} - 1$ iteration. For random variables in a linear system, this can be written in the form

$$\mathbf{y}_k = \mathbf{F}_{k-1}\mathbf{y}_{k-1} + \mathbf{w}_{k-1}, \quad \forall k \in \mathbb{Z}_+ \setminus \{0\} \tag{17}$$

where $\mathbf{w}_{k-1}$ is the noise associated with the model. For a linear multidisciplinary design, Eq. (17) can also be written as

$$\mathbf{y}_k = \mathbf{F}_{k-1}\mathbf{y}_{k-1} + \mathbf{B}_{k-1}\mathbf{u}_{k-1} + \mathbf{w}_{k-1} \tag{18}$$

which allows for inputs into the CA that are not outputs of other CAs, $\mathbf{u}_{k-1}$. When coupled with an equation of the form

$$\mathbf{z}_k = \mathbf{H}_{k-1}\mathbf{y}_{k-1} + \mathbf{v}_{k-1} \tag{19}$$

and when it is assumed that $\mathbf{w}_{k-1} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_{k-1})$ and $\mathbf{v}_{k-1} \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_{k-1})$, Eqs. (18) and (19) define the dynamical system needed for a Kalman filter.[56–59] The noise parameter, $\mathbf{w}_{k-1}$, gives the opportunity to account for random variables within the linearization of the input-output relationship, that is random variables associated with the matrix $\mathbf{F}$. In this work, the Kalman filter is used as a data fusion technique to give an optimal unbiased statistical estimate of the output of the CAs as the design is converging.

The Kalman filter can be thought of as a two step process, one which predicts the state (*e.g.*, the output of the CAs) and then an update step which corrects these estimates based on the dynamics of the system. The prediction step is given by the following equations[39, 56–61]

$$\hat{\mathbf{y}}_{k|k-1} = \mathbf{F}_k\hat{\mathbf{y}}_{k-1|k-1} + \mathbf{B}_k\mathbf{u}_k \tag{20}$$

$$\mathbf{\Sigma}_{k|k-1} = \mathbf{F}_k\mathbf{\Sigma}_{k-1|k-1}\mathbf{F}_k^T + \mathbf{Q}_k \tag{21}$$

where the notation $j|k$ represents the estimate at j given observations up to and including k. Furthermore, the value of $\hat{\mathbf{y}}_{0|0}$ is the initial mean state and $\mathbf{\Sigma}_{0|0}$ is the initial covariance matrix of the state values. The correction step is governed by the following equations[39, 56–61]

$$\tilde{\mathbf{x}}_k = \mathbf{z}_k - \mathbf{H}_k\hat{\mathbf{y}}_{k|k-1} \tag{22}$$

$$\mathbf{S}_k = \mathbf{H}_k\mathbf{\Sigma}_{k|k-1}\mathbf{H}_k^T + \mathbf{R}_k \tag{23}$$

$$\mathbf{K}_k = \mathbf{\Sigma}_{k|k-1}\mathbf{H}_k^T\mathbf{S}_k^{-1} \tag{24}$$

$$\hat{\mathbf{y}}_{k|k} = \hat{\mathbf{y}}_{k|k-1} + \mathbf{K}_k\tilde{\mathbf{x}}_k \tag{25}$$

$$\mathbf{\Sigma}_{k|k} = (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\,\mathbf{\Sigma}_{k|k-1} \tag{26}$$

where the final (*a posteriori*) estimate of the state is given by $\hat{\mathbf{y}}_{k|k}$ with covariance matrix given by $\mathbf{\Sigma}_{k|k}$.

The power in implementing the Kalman filter in multidisciplinary design analysis lies in the ability to obtain a continuous estimate in iterate of both the mean and covariance of each CA in the multidisciplinary design by propagating a system of seven equations until the design converges.

II.G. Matrix Norms

The output of the Kalman filter is the estimated mean and covariance matrix given by

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma^2_{X_1} & \rho_{X_1,X_2}\sigma_{X_1}\sigma_{X_2} & \cdots & \rho_{X_1,X_2}\sigma_{X_1}\sigma_{X_n} \\ \rho_{X_1,X_2}\sigma_{X_2}\sigma_{X_1} & \sigma^2_{X_2} & \cdots & \rho_{X_2,X_n}\sigma_{X_2}\sigma_{X_n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{X_1,X_n}\sigma_{X_n}\sigma_{X_1} & \rho_{X_1,X_n}\sigma_{X_n}\sigma_{X_2} & \cdots & \sigma^2_{X_n} \end{pmatrix} \tag{27}$$

where $\sigma^2_{X_i}$ is the variance of variable X_i and ρ_{X_i,X_j} is the product-moment coefficient (*i.e.*, the correlation coefficient) given by

$$\rho_{X_i,X_j} = \frac{\mathbb{E}\left[(X_i - \mu_{X_i})(X_j - \mu_{X_j})\right]}{\sigma_{X_i}\sigma_{X_j}} \tag{28}$$

In order to obtain a singular value that bounds the variance, consider the matrix 2-norm which is defined as

$$\parallel \mathbf{A} \parallel_2 = \sqrt{\lambda_{\max}(\mathbf{A}^*\mathbf{A})} \tag{29}$$

where $\mathbf{A}^*$ represents the conjugate transpose of a matrix $\mathbf{A}$ and $\lambda_{\max}(\cdot)$ is a function which returns the maximum eigenvalue. The 2-norm can be more readily understood in the context of spectral decomposition such that $\mathbf{D} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$ where $\mathbf{D}$ is at worst a block-diagonal matrix. In the case of real, distinct eigenvalues, the diagonal of matrix $\mathbf{D}$ consists of the eigenvalues. By virtue of the properties of the covariance matrix, $\lambda_{\max}(\mathbf{\Sigma}) \geq \sigma^2_{\max}$, which means that the 2-norm provides a bound on the variance.

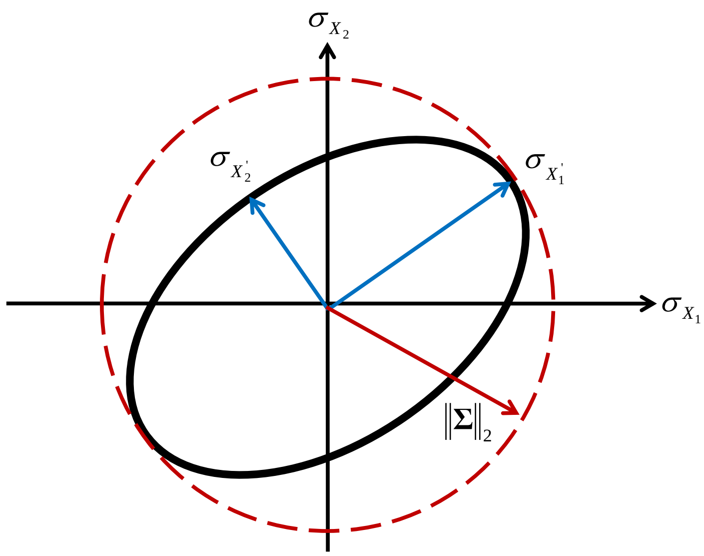


Figure 5. Visual representation of the matrix 2-norm.

In two-dimensions, this can be seen in Fig. 5 where the covariance matrix is plotted as an ellipse. In Fig. 5, the axes $\sigma_{X_1}\sigma_{X_2}$ are the standard deviations associated with the covariance matrix, $\mathbf{\Sigma}$. The eigenvectors of the covariance matrix form the alternate set of axes (in blue), $\sigma_{X_1'}\sigma_{X_2'}$. The 2-norm is the variance of the "pseudo-variable" that is oriented along the principal eigenvector of the resulting ellipse, which is the magnitude of the semi-major axis of the ellipse. In other words, the 2-norm is the radius of the circle which completely encompasses the covariance matrix. An advantageous feature of this norm is that it is always a conservative estimate of the variance of the system. Furthermore, as the dimensionality of the problem increase, this overestimate diminishes due to the 2-norm acting similarly to a root-sum-square (or Euclidean norm).

III. A Rapid Robust Multidisciplinary Design Methodology Using Dynamical System Theory

The following section describes a matrix-norm bound technique to obtain an estimate of the mean and a bound on the variance of a multidisciplinary system which can be decomposed into CAs.

Step 1: Decompose the Design
A general multidisciplinary design can be decomposed into multiple CAs. Each of these CAs represents an analysis that contributes to the entire design. For example, consider the design or analysis of an entry system. It may be desired for the entry system to be evaluated with respect to its payload capability and landing accuracy. Many different analyses must be conducted in order to obtain this information. This information flow is shown in Fig. 6, where one such representation of each of the analyses that must be conducted to design an entry system is shown.[62]

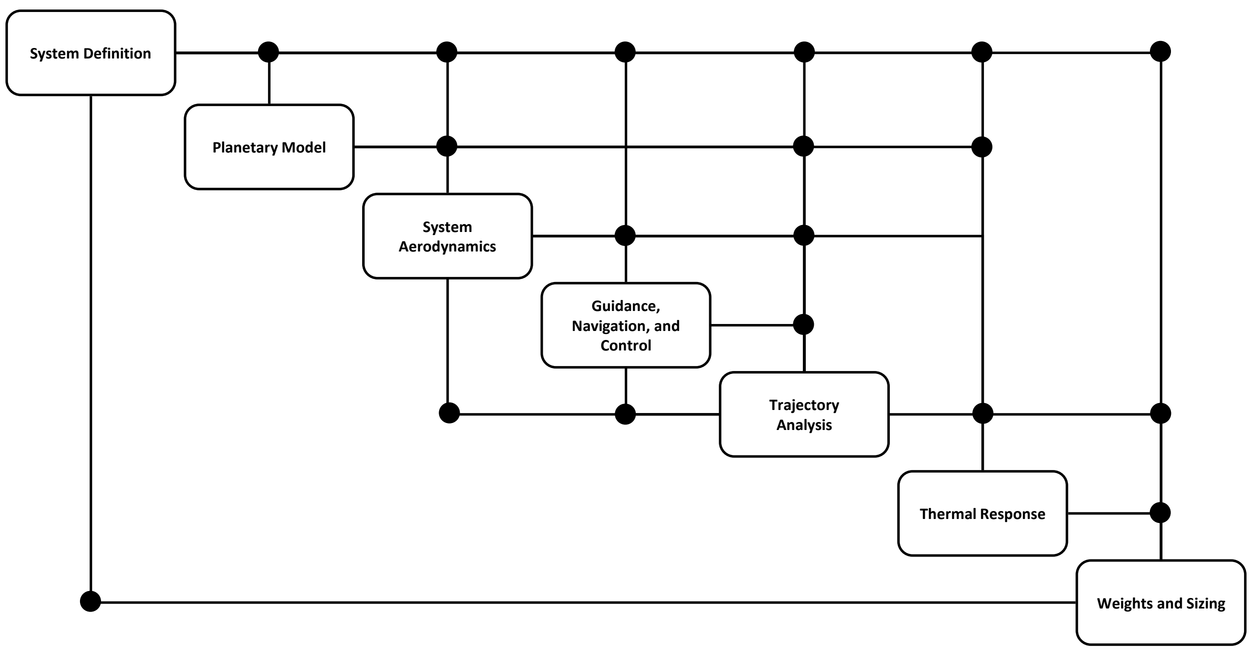


Figure 6. The decomposition of an entry system into a Design Structure Matrix.

In this case, the entry system is decomposed into the seven CAs. The cumulative sum of these CAs allow the payload capability as well as the landed accuracy to be assessed. Each CA in Fig. 6 (*i.e.*, the blocks) represents an input-output relationship. For instance, inputs into the aerodynamics analysis include the configuration of the entry system and planetary body where it is to operate; outputs include the force coefficients of the vehicle as a function of Mach number and attitude. This relationship may be known analytically; however, it is more likely that this CA would represent a computational analysis that is linked into the design process.

In the theoretical development underlying this work, it is assumed that each of the n CAs are linear and algebraic. This limitation will be addressed subsequently. That is, the output of the CA is of the form

$$\mathbf{y_j} = \mathbf{A_j}\mathbf{y} + \mathbf{B_j}\mathbf{u_d} + \mathbf{C_j}\mathbf{u_p} + \mathbf{d_j} \tag{30}$$

where $\mathbf{y_j} \in \mathbb{R}^{l_j}$, $\mathbf{y} \in \mathbb{R}^m$ is the concatenated output from all of the CAs (*e.g.*, if $\mathbf{y_1}$, $\mathbf{y_2}$ through $\mathbf{y_n}$ are the outputs of the n CAs in a multidisciplinary design, $\mathbf{y} = \left(\mathbf{y_1}^T \ \mathbf{y_2}^T \ \cdots \ \mathbf{y_n}^T\right)^T$), $\mathbf{u_d} \in \mathbb{R}^d$ are the deterministic system-level inputs into the design, $\mathbf{u_p} \in \mathbb{R}^p$ are the probabilistic system-level inputs into the design, and $\mathbf{d_j} \in \mathbb{R}^{l_j}$ is the bias associated with the model. This implies $\mathbf{A_j} \in \mathbb{R}^{l_j \times m}$, $\mathbf{B_j} \in \mathbb{R}^{l_j \times d}$, $\mathbf{C_j} \in \mathbb{R}^{l_j \times p}$, and that $\sum_{j=1}^{n} l_j = m$.

For general designs where the CAs may not be linear, the required functional form can be achieved through linearization where $\mathbf{A_j} = \left.\frac{\partial \mathbf{g}}{\partial \mathbf{y}}\right|_{\tilde{\mathbf{y}}}$, $\mathbf{B_j} = \left.\frac{\partial \mathbf{g}}{\partial \mathbf{u_d}}\right|_{\tilde{\mathbf{u}}_\mathbf{d}}$, $\mathbf{C_j} = \left.\frac{\partial \mathbf{g}}{\partial \mathbf{u_p}}\right|_{\tilde{\mathbf{u}}_\mathbf{p}}$, and $\mathbf{d_j} = -\left(\mathbf{A_j}\tilde{\mathbf{y}} + \mathbf{B_j}\tilde{\mathbf{u}}_\mathbf{d} + \mathbf{C_j}\tilde{\mathbf{u}}_\mathbf{p}\right)$ when the input-output relationship for the CA is given by $\mathbf{y_j} = \mathbf{g}\left(\mathbf{y}, \mathbf{u_d}, \mathbf{u_p}\right)$ and $\tilde{(\cdot)}$ is the value of $(\cdot)$ about which the function is linearized.

Step 2: Identify the Random Variables and their Distributions

In the design and analysis of a complex multidisciplinary system, it is unlikely that each of the models and inputs are deterministic; instead, many are likely to be probabilistic and account for unknowns in the modeling and in the operating conditions . In order to propagate these uncertainties through the design to estimate the robustness, the probabilistic variables must be identified.

The random variables associated with the uncertainty within the design are handled in two different ways in this work depending on where the random variable is functionally located. Functionally, the uncertainty resulting from inputs into the CA refers to uncertainties associated with $\mathbf{u_p}$, whereas uncertainty associated with the physical modeling pertain to $\mathbf{A_j}$, $\mathbf{B_j}$, $\mathbf{C_j}$, or $\mathbf{d_j}$. In the first instance, the mean is propagated in the $\hat{\mathbf{y}}_{k|k}$ term of the filter equations and the covariance is propagated in the $\mathbf{\Sigma}_{k|k}$ term of the filter equations. In the second case, the mean is again accounted for in the $\hat{\mathbf{y}}_{k|k}$ term of the equations; however, the covariance is accounted for in the $\mathbf{Q}_k$ term of the filter. In the components of the Kalman filter mentioned, the notation $k|k$ refers to the k^{th} iteration of the filter given all previous information regarding the convergence of the system.

Due to the the propagation within the Kalman filter there is an assumption that the uncertainties associated with the model are Gaussian. For symmetrical probability distributions (*i.e.*, probability distributions centered about the mean), this is not an overly strong assumption since the first two moments are the only terms being approximated. However, for asymmetric probability distributions, this becomes a restrictive assumption that is a limitation of the proposed technique.

Step 3: Form the Iterative Equations

The process of converging the multidisciplinary design through root solving leads to an inherent dynamical system. This root can be sought out using an iterative technique. For example, fixed-point iteration, defined in Eq. (16) uses the previous iteration's solution as an input to the current iteration. For this work, $\mathbf{f}(\cdot)$ is the concatenation of the input-output relationships for the CAs (*e.g.*, if $\mathbf{f_1}(\cdot)$, $\mathbf{f_2}(\cdot)$ through $\mathbf{f_n}(\cdot)$ describe the input-output relationship for each of the n CAs, $\mathbf{f}(\cdot) = \begin{pmatrix}\mathbf{f_1}^T(\cdot) & \mathbf{f_2}^T(\cdot) & \cdots & \mathbf{f_n}^T(\cdot)\end{pmatrix}^T$). In the framework described here, where the multidisciplinary design consists solely of linear CAs, the fixed-point iteration relationship becomes extremely tractable

$$\mathbf{y}_k = \mathbf{\Lambda}\mathbf{y}_{k-1} + \boldsymbol{\beta}\mathbf{u_d} + \boldsymbol{\gamma}\mathbf{u_p} + \boldsymbol{\delta} \tag{31}$$

where it is assumed that $\mathbf{\Lambda} = \begin{pmatrix}\mathbf{A_1}\\ \vdots \\ \mathbf{A_n}\end{pmatrix} \in \mathbb{R}^{m\times m}$, $\boldsymbol{\beta} = \begin{pmatrix}\mathbf{B_1}\\ \vdots \\ \mathbf{B_n}\end{pmatrix} \in \mathbb{R}^{m\times d}$, $\boldsymbol{\gamma} = \begin{pmatrix}\mathbf{C_1}\\ \vdots \\ \mathbf{C_n}\end{pmatrix} \in \mathbb{R}^{m\times p}$, and $\boldsymbol{\delta} = \begin{pmatrix}\mathbf{d_1}\\ \vdots \\ \mathbf{d_n}\end{pmatrix} \in \mathbb{R}^{m}$.

Step 4: Ensure a Solution Exists

Since the iterative system defined by Eq. (31) is a discrete, linear, dynamical system, the existence of a solution to the multidisciplinary design problem is given solely by the stability of the system. In particular, if the system is asymptotically stable, a converged design exists for some initial guess of the CA outputs and if it is globally asymptotically stable, a design exists for all initial guesses of the CA outputs.

For cases where $\mathbf{\Lambda}$ is a constant matrix, finding the eigenvalues of the matrix $\mathbf{\Lambda}$ determines the existence of a design solution. Should all of these eigenvalues have modulus less than unity (*i.e.*, $|\lambda_i| < 1$) the dynamical system is globally asymptotically stable and the multidisciplinary system will converge regardless of the initial guess for the output of the CAs. However, should this not be the case, and at least one eigenvalue has modulus greater than or equal to unity (*i.e.*, $|\lambda_i| \geq 1$), a contraction mapping does not exist for the choice of root-finding schemes and the design will not converge.

When $\mathbf{\Lambda}$ is a varying matrix or even a nonlinear mapping, a Lyapunov function technique can be used to investigate the stability (and convergence) of the design. In this case, for asymptotic stability, a positive-definite function is sought whose difference between iterates in some region around the origin is negative definite. The search for a Lyapunov function can be accomplished using several methods, including some numerical based techniques (see Refs. 10 and 63).

Step 5: Estimate the Mean Output and the Covariance

The mean output of the multidisciplinary system and the associated covariance matrix are found by propagating the Kalman filter equations, Eqs. (20)-(26) until convergence. In order to accomplish this, the iterative system formed in Eq. (31) needs to be transformed to the form needed in Kalman filter, Eq. (18). This is a relatively straightforward process when the following substitutions are made

$$\mathbf{F}_{k-1} = \mathbf{\Lambda}, \quad \forall k \in \{1, 2, ..., n\} \tag{32}$$

$$\mathbf{B}_{k-1} = \begin{pmatrix} \boldsymbol{\beta} & \boldsymbol{\gamma} & \mathbf{I_{m\times m}} \end{pmatrix}, \quad \forall k \in \{1, 2, ..., n\} \tag{33}$$

$$\mathbf{u}_{k-1} = \begin{pmatrix} \mathbf{u_d} \\ \mathbf{u_p} \\ \boldsymbol{\delta} \end{pmatrix}, \quad \forall k \in \{1, 2, ..., n\} \tag{34}$$

The mean state, that is the output of the analyses, $(\hat{\mathbf{y}}_{0|0})$ and the covariance matrix associated with the state $(\mathbf{\Sigma}_{0|0})$ are initialized by the relations

$$\hat{\mathbf{y}}_{0|0} = \mathbf{y_0} \tag{35}$$

$$\mathbf{\Sigma}_{0|0} = \mathbf{\Sigma_0} \tag{36}$$

In this work, $\mathbf{y_0}$ and $\mathbf{\Sigma_0}$ are found by assuming a starting value for the coupled CA and an input covariance matrix associated with the parameters of the problem. These values are then propagated through each CA of a serial (*i.e.*, uncoupled) design structure matrix using the unscented transform technique. The concatenated output of each of the CAs is then used to form $\mathbf{y_0}$ and the covariance matrix $\mathbf{\Sigma_0}$, which will initially be a block diagonal matrix. The last parameter which need to be identified in order to estimate the mean output and the covariance of the system is the covariance matrix associated with the model, $\mathbf{Q}$. This is a block diagonal matrix composed of the variances and covariances associated with $\mathbf{A_j}$, $\mathbf{B_j}$, $\mathbf{C_j}$, and $\mathbf{d_j}$.

The iterates are then found by by propagating the filter equations, Eqs. (20)-(26), with $\mathbf{H}_{k-1} = \mathbf{I_{m\times m}} \quad \forall k \in \{1, 2, ..., n\}$ and $\mathbf{R}_{k-1} = \mathbf{0} \quad \forall k \in \{1, 2, ..., n\}$ until the design convergence criterion is met. The exact convergence criterion can be of several forms, the two criterion used within this work are an absolute tolerance of the state and a relative tolerance of the state. These are demonstrated in the following relations

$$\| \hat{\mathbf{y}}_{k|k} - \hat{\mathbf{y}}_{k-1|k-1} \|_2 \leq \epsilon_1 \tag{37}$$

$$\frac{\| \hat{\mathbf{y}}_{k|k} - \hat{\mathbf{y}}_{k-1|k-1} \|_2}{\| \hat{\mathbf{y}}_{k-1|k-1} \|_2} \leq \epsilon_2 \tag{38}$$

Step 6: Identify the Mean and Variance Bound of the Objective Function

Assume the design objective is a linear combination of the outputs of linear CAs, that is

$$r = \mathbf{M}\mathbf{y}^* \tag{39}$$

where $r \in \mathbb{R}$ is the design objective value, $\mathbf{M} \in \mathbb{R}^{1\times q}$ is a matrix describing the linear combination of the pertinent CA outputs, and $\mathbf{y}^* \in \mathbb{R}^q$ is the vector of pertinent CA responses that contribute to the design objective. An estimate of the mean and variance bound for the design objective can be found as follows

$$\bar{r} = \mathbf{M}\hat{\mathbf{y}}^*_{n|n} \tag{40}$$

$$\sigma_r^2 \leq \| \mathbf{\Sigma}_{\boldsymbol{y^*}\, n|n} \|_2 \, \mathbf{M}\mathbf{1}_q \tag{41}$$

where it is assumed the n iterations have occurred and $\mathbf{\Sigma}_{\boldsymbol{y^*}_{n|n}}$ is the reduced covariance matrix associated with only the variables associated with $\mathbf{y}^*$ (*i.e.*, the rows and columns of the variables not pertinent in the design objective are eliminated from $\mathbf{\Sigma}_{\boldsymbol{y}_{n|n}}$). Additionally, the notation $\mathbf{1}_q \in \mathbb{R}^{q\times 1}$ is the unity vector of length q (*i.e.*, $\mathbf{1}_q = (1\ 1\ 1 \cdots\ 1)^T \in \mathbb{R}^{q\times 1}$).

More generally, a first-order expansion of an objective function that is of the form

$$r = g(\mathbf{y}^*) \tag{42}$$

can be made. The linearized objective function, Eq. (42), about $\mathbf{y}^*_{\mathbf{nom}}$ is then given by

$$\tilde{r} = \frac{\partial g}{\partial \mathbf{y}^*}\mathbf{y}^* - \frac{\partial g}{\partial \mathbf{y}^*}\mathbf{y}^*_{\mathbf{nom}} = \mathbf{N}\mathbf{y}^* + b \tag{43}$$

which leads to the results

$$\bar{r} \approx \mathbf{N}\hat{\mathbf{y}}^*_{n|n} + b \tag{44}$$

$$\sigma_r^2 \approx \| \mathbf{\Sigma}_{\boldsymbol{y^*}_{n|n}} \|_2\ \mathbf{N}\mathbf{1}_q^T \tag{45}$$

where it is assumed that $\mathbf{N} \in \mathbb{R}^{1\times q}$.

Step 7: Optimize for Uncertainty and Ensure Constraints are Met
Formulating the output of Step 6 in terms of the mean and variance allows for an optimal control problem to be setup where the objective function is defined by

$$\mathcal{J} = \mathbf{M}\left(\alpha\hat{\mathbf{y}}^*_{n|n} + \beta \parallel \mathbf{\Sigma}_{\boldsymbol{y}^*_{\boldsymbol{n|n}}} \parallel_2 \mathbf{1}_q^T\right) \tag{46}$$

and α and β are weights on the relative components that can be varied to find different compromised optimal designs. The problem is then to seek out the control, $\mathbf{u}$, that minimizes $\mathcal{J}$. In this case the control is constant (since they are parameters of the problem) and given by $\mathbf{u_d}$. The requirements outside of the compatibility constraints are then handled by adjoining the set of convex constraints to the objective function and identifying an optimum that satisfies the necessary conditions outlined previously.

Step 8: Evaluate the Quality of the Robustness Estimate
The quality of the robustness estimate can be evaluated by using the unscented transform to get a higher-order estimate of the mean and the covariance of the output. This step, however, may be time consuming and may not be desirable to perform in all instances, particularly if the design is known to be linear, as the propagation by the Kalman filter through a linear system is exact. The procedure to obtain this estimate is as follows:

1. Identify the uncertain parameters for the problem and form the initial covariance matrix for these parameters
2. Identify the m (or $m + 1$ if an alternate form of the unscented transform is used) sigma points based on the eigenstructure of the initial covariance matrix
3. Propagate each of these sigma points through the design until convergence
4. Record the objective function for each sigma point propagation
5. Compute the scalar mean and variance from the composite results for each of the objective function values

III.A. Validity of a Linear Technique

The theoretical development of the rapid robust design methodology as a complete ensemble is restricted to linear systems; however, the individual methods as well as the complete the methods employed in the complete ensemble are inherently extensible to nonlinear systems. Specifically, the Kalman Filter can be replaced with a filter designed for nonlinear estimation (*e.g.*, the Extended Kalman Filter, the Unscented Kalman Filter, or a Particle Filter) and the unscented transform is designed for nonlinear propagation of random variables.

IV. Accuracy and Performance of the Rapid Robustness Assessment Methodology

To show the accuracy of the mean and variance estimate provided by the proposed methodology, consider the coupled, linear two CA system shown in Fig. 7.

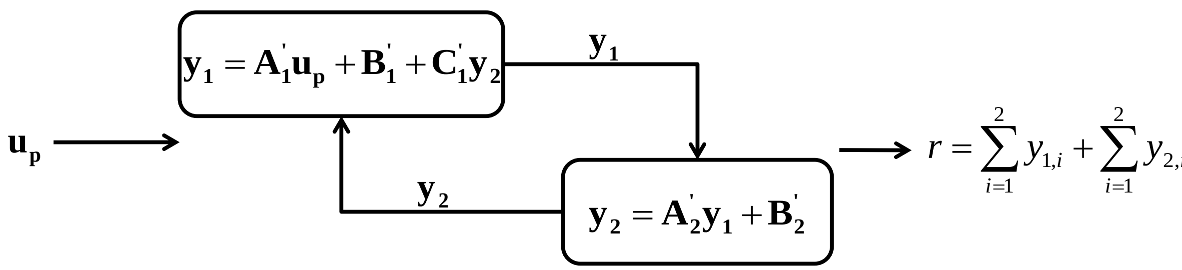


Figure 7. Two-contributing analysis multidisciplinary design.

For this analysis, assume that there are two components to the probabilistic parameter vector and the two output vectors, that is $\mathbf{u_p} \in \mathbb{R}^2$, $\mathbf{y_1} \in \mathbb{R}^2$, and $\mathbf{y_2} \in \mathbb{R}^2$, which, in turn, implies $\mathbf{A_1'} \in \mathbb{R}^{2\times2}$, $\mathbf{B_1'} \in \mathbb{R}^2$, $\mathbf{C_1'} \in \mathbb{R}^{2\times2}$, $\mathbf{A_2'} \in \mathbb{R}^{2\times2}$, and $\mathbf{B_2'} \in \mathbb{R}^2$. Also, let the distribution of the probabilistic parameter input be given by a multivariate normal, $\mathbf{u_p} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{u_p}}, \boldsymbol{\Sigma}_{\mathbf{u_p}})$.

The effectiveness of the robustness assessment methodology will be demonstrated by letting the mean of the probabilistic input, $\boldsymbol{\mu}_{\mathbf{u_p}}$, and the components of the covariance matrix $\sigma^2_{y_1}$, $\sigma^2_{y_2}$, and $\rho_{y_1 y_2}$ vary between given ranges. The maximum error between the response obtained from the robustness assessment methodology and an analytical propagation is then reported for a multitude of points within the design space.

IV.A. Analytical Solution

As this is a multidisciplinary analysis consisting of two linear CAs, there is a single simultaneous solution for $\mathbf{y_1}$ and $\mathbf{y_2}$ which is found to be

$$\left.\begin{array}{rcl} \mathbf{y_1} &=& \left(\mathbf{I_{2\times2}} - \mathbf{C_1'A_2'}\right)^{-1}\left(\mathbf{A_1'u_p} + \mathbf{B_1'} + \mathbf{C_1'B_2'}\right) \\ \mathbf{y_2} &=& \mathbf{A_2'}\left(\mathbf{I_{2\times2}} - \mathbf{C_1'A_2'}\right)^{-1}\left(\mathbf{A_1'u_p} + \mathbf{B_1'} + \mathbf{C_1'B_2'}\right) + \mathbf{B_2'} \end{array}\right\}$$

which implies that whenever $\mathbf{I_{2\times2}} - \mathbf{C_1'A_2'}$ is non-singular, a unique solution exists for $\mathbf{y_1}$ and $\mathbf{y_2}$. Since the only uncertainty in this analysis is given by the probabilistic input vector, $\mathbf{u_p}$, which is defined as a multivariate normal, the distribution of the output for each CA can be found exactly. These are given by

$$\left.\begin{array}{rcl} \mathbf{y_1} &\sim& \mathcal{N}\left(\boldsymbol{\mu}_{\mathbf{y_1}}, \boldsymbol{\Sigma}_{\mathbf{y_1}}\right) \\ \mathbf{y_2} &\sim& \mathcal{N}\left(\boldsymbol{\mu}_{\mathbf{y_2}}, \boldsymbol{\Sigma}_{\mathbf{y_2}}\right) \end{array}\right\}$$

where

$$\begin{array}{rcl} \boldsymbol{\mu}_{\mathbf{y_1}} &=& \left(\mathbf{I_{2\times2}} - \mathbf{C_1'A_2'}\right)^{-1}\mathbf{A_1'}\boldsymbol{\mu}_{\mathbf{u_p}} + \left(\mathbf{I_{2\times2}} - \mathbf{C_1'A_2'}\right)^{-1}\left(\mathbf{B_1'} + \mathbf{C_1'B_2'}\right) \\ \boldsymbol{\Sigma}_{\mathbf{y_1}} &=& \left(\mathbf{I_{2\times2}} - \mathbf{C_1'A_2'}\right)^{-1}\mathbf{A_1'}\boldsymbol{\Sigma}_{\mathbf{u_p}}\mathbf{A_1'}^T\left(\mathbf{I_{2\times2}} - \mathbf{C_1'A_2'}\right)^{-T} \end{array}$$

and

$$\begin{array}{rcl} \boldsymbol{\mu}_{\mathbf{y_2}} &=& \mathbf{A_2'}\left(\mathbf{I_{2\times2}} - \mathbf{C_1'A_2'}\right)^{-1}\mathbf{A_1'}\boldsymbol{\mu}_{\mathbf{u_p}} + \mathbf{A_2'}\left(\mathbf{I_{2\times2}} - \mathbf{C_1'A_2'}\right)^{-1}\left(\mathbf{B_1'} + \mathbf{C_1'B_2'}\right) + \mathbf{B_2'} \\ \boldsymbol{\Sigma}_{\mathbf{y_2}} &=& \mathbf{A_2'}\left(\mathbf{I_{2\times2}} - \mathbf{C_1'A_2'}\right)^{-1}\mathbf{A_1'}\boldsymbol{\Sigma}_{\mathbf{u_p}}\mathbf{A_1'}^T\left(\mathbf{I_{2\times2}} - \mathbf{C_1'A_2'}\right)^{-T}\mathbf{A_2'}^T \end{array}$$

Since both of the output distributions from the CAs are also multivariate normal, the components of the response

$$r = \sum_{i=1}^{2} y_{1,i} + \sum_{i=1}^{2} y_{2,i}$$

can be found exactly by summing the components of mean components of $\boldsymbol{\mu}_{\mathbf{y_1}}$ and $\boldsymbol{\mu}_{\mathbf{y_2}}$ to find the mean of the response and adding the appropriate variances from the covariance matrices $\boldsymbol{\Sigma}_{\mathbf{y_1}}$ and $\boldsymbol{\Sigma}_{\mathbf{y_2}}$. That is

$$r \sim \mathcal{N}\left(\sum_{i=1}^{2}\boldsymbol{\mu}_{\mathbf{y_1},i} + \sum_{i=1}^{2}\boldsymbol{\mu}_{\mathbf{y_2},i}, \sum_{i=1}^{2}\lambda(\boldsymbol{\Sigma}_{\mathbf{y_1}})|_i + \sum_{i=1}^{2}\lambda(\boldsymbol{\Sigma}_{\mathbf{y_2}})|_i\right)$$

where $\boldsymbol{\mu}_{\mathbf{y_1},i}$ is the i^{th} component of $\mathbf{y_1}$, $\boldsymbol{\mu}_{\mathbf{y_2},i}$ is the i^{th} component of $\mathbf{y_2}$, and $\lambda(\cdot)|_i$ is the i^{th} eigenvalue of the matrix argument.

IV.B. Rapid Robustness Assessment Methodology

In assessing the robustness methodology, the first five steps of the developed methodology will be followed. These steps obtain an estimate of the output mean and a bound on the variance provided by the 2-norm of the covariance matrix obtained by propagating the dynamical system through a Kalman filter.

Step 1: Decompose the Design

The problem as given has already been decomposed into the representative contributing analyses; however, it is still necessary to identify each of the terms in Eq. (30). For the first CA, $\mathbf{y_1}$, the functional form of the CA is as follows

$$\mathbf{y_1} = \begin{pmatrix}\mathbf{0} & \mathbf{C}'_\mathbf{1}\end{pmatrix}\mathbf{y} + \begin{pmatrix}\mathbf{0}\end{pmatrix}\mathbf{u_d} + \begin{pmatrix}\mathbf{A}'_\mathbf{1}\end{pmatrix}\mathbf{u_p} + \mathbf{B}'_\mathbf{1}$$

Similarly, for the second CA, the functional form is given by

$$\mathbf{y_2} = \begin{pmatrix}\mathbf{A}'_\mathbf{2} & \mathbf{0}\end{pmatrix}\mathbf{y} + \begin{pmatrix}\mathbf{0}\end{pmatrix}\mathbf{u_d} + \begin{pmatrix}\mathbf{0}\end{pmatrix}\mathbf{u_p} + \mathbf{B}'_\mathbf{2}$$

Hence,

$$\mathbf{A_1} = \begin{pmatrix}\mathbf{0} & \mathbf{C}'_\mathbf{1}\end{pmatrix} \quad \mathbf{B_1} = \begin{pmatrix}\mathbf{0}\end{pmatrix}$$

$$\mathbf{C_1} = \begin{pmatrix}\mathbf{A}'_\mathbf{1}\end{pmatrix} \quad \mathbf{d_1} = \mathbf{B}'_\mathbf{1}$$

$$\mathbf{A_2} = \begin{pmatrix}\mathbf{A}'_\mathbf{2} & \mathbf{0}\end{pmatrix} \quad \mathbf{B_2} = \begin{pmatrix}\mathbf{0}\end{pmatrix}$$

$$\mathbf{C_2} = \begin{pmatrix}\mathbf{0}\end{pmatrix} \quad \mathbf{d_2} = \mathbf{B}'_\mathbf{2}$$

Step 2: Identify the Random Variables and their Distributions

There is only one set of random variables in this example, that of the probabilistic input variable, $\mathbf{u_p}$. This is given in the problem description as a multivariate normal distribution, $\mathbf{u_p} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{u_p}}, \boldsymbol{\Sigma}_{\mathbf{u_p}})$. In this case the covariance matrix is given by

$$\boldsymbol{\Sigma}_{\mathbf{u_p}} = \begin{pmatrix}\sigma^2_{u_{p,1}} & \rho_{u_{p,1},u_{p,2}}\sigma_{u_{p,1}}\sigma_{u_{p,2}} \\ \rho_{u_{p,1},u_{p,2}}\sigma_{u_{p,1}}\sigma_{u_{p,2}} & \sigma^2_{u_{p,2}}\end{pmatrix}$$

Later, the defining parameters of the multivariate normal will be given numerical values.

Step 3: Form the Iterative Equations

In order to use the Kalman filter to simultaneously estimate the robustness and converge the design, the iterative equations described in Eq. (31) for fixed-point iteration need to be formed. Through analogy of variables, the matrices are given by

$$\boldsymbol{\Lambda} = \begin{pmatrix}\mathbf{A_1} \\ \mathbf{A_2}\end{pmatrix} = \begin{pmatrix}\mathbf{0} & \mathbf{C}'_\mathbf{1} \\ \mathbf{A}'_\mathbf{2} & \mathbf{0}\end{pmatrix}$$

$$\boldsymbol{\beta} = \begin{pmatrix} \mathbf{B_1} \\ \mathbf{B_2} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{\gamma} = \begin{pmatrix} \mathbf{C_1} \\ \mathbf{C_2} \end{pmatrix} = \begin{pmatrix} \mathbf{A_1'} \\ \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{\delta} = \begin{pmatrix} \mathbf{d_1} \\ \mathbf{d_2} \end{pmatrix} = \begin{pmatrix} \mathbf{B_1'} \\ \mathbf{B_2'} \end{pmatrix}$$

Step 4: Ensure a Solution Exists

In this problem, the variables are yet to be defined. However, they are constant coefficients. This implies that the likelihood of finding a solution is dependent entirely on the matrix $\boldsymbol{\Lambda}$, providing a constraint to the values which will be examined in this design space analysis. Expanding $\boldsymbol{\Lambda}$ allows the characteristic equation to be found

$$\boldsymbol{\Lambda} = \begin{pmatrix} 0 & 0 & C_1 & C_2 \\ 0 & 0 & C_3 & C_4 \\ A_1 & A_2 & 0 & 0 \\ A_3 & A_4 & 0 & 0 \end{pmatrix}$$

Therefore, the characteristic equation is given by

$$\det(\boldsymbol{\Lambda} - \lambda \mathbf{I_{4\times4}}) = \begin{vmatrix} -\lambda & 0 & C_1 & C_2 \\ 0 & -\lambda & C_3 & C_4 \\ A_1 & A_2 & -\lambda & 0 \\ A_3 & A_4 & 0 & -\lambda \end{vmatrix} = 0$$

which can be solved in order to ensure that the modulus of each of the eigenvalues is less than one for repeated roots or less than or equal to one for simple roots.

Step 5: Estimate the Mean Output and the Covariance

The equations formed in the prior step can then be propagated through the Kalman filter defined by Eqs. (20)-(26) with

$$\mathbf{F}_{k-1} = \boldsymbol{\Lambda}, \quad \forall k \in \{1, 2, ..., n\}$$

$$\mathbf{B}_{k-1} = \begin{pmatrix} \boldsymbol{\beta} & \boldsymbol{\gamma} & \mathbf{I_{4\times4}} \end{pmatrix}, \quad \forall k \in \{1, 2, ..., n\}$$

$$\mathbf{u}_{k-1} = \begin{pmatrix} \mathbf{u_d} \\ \mathbf{u_p} \\ \boldsymbol{\delta} \end{pmatrix}, \quad \forall k \in \{1, 2, ..., n\}$$

where in this example $\mathbf{u_d} = \mathbf{0}$ and $\mathbf{u_p} = \mathbb{E}(\mathbf{u_p}) = \boldsymbol{\mu}_{\mathbf{u_p}}$. In this example, the matrix $\mathbf{Q}$ is the null matrix since the only uncertain parameters of the problem are associated with the input parameters, not the model. The unscented transform is used on an uncoupled system with the distribution described in Step 2 in order to identify $\mathbf{y_0}$ and $\boldsymbol{\Sigma_0}$, the initial output mean and covariance for each design. A design is considered converged when the absolute difference between iteration estimates is less than 1×10^{-4} or the relative difference is less than 1×10^{-6}.

IV.C. Analysis Results

In order to assess a large variety of problems, a parametric sweep of the design variables was performed to identify the maximum errors in the design space. To perform this parameter sweep, the problem's parameters were varied independently as shown in Table 5 where the distribution of each variable was assumed to be uniform and a 100,000 case Monte Carlo analysis was conducted. In order to guarantee convergence of the design, constraints were imposed on the parameters to ensure that all of the eigenvalues of the matrix $\boldsymbol{\Lambda}$ had modulus less than unity. To ensure realizable covariance matrices, that is a matrix that is symmetric and positive definite, the components of the covariance (*e.g.*, variance and correlation coefficient) were determined independently and then combined to form the covariance matrix.

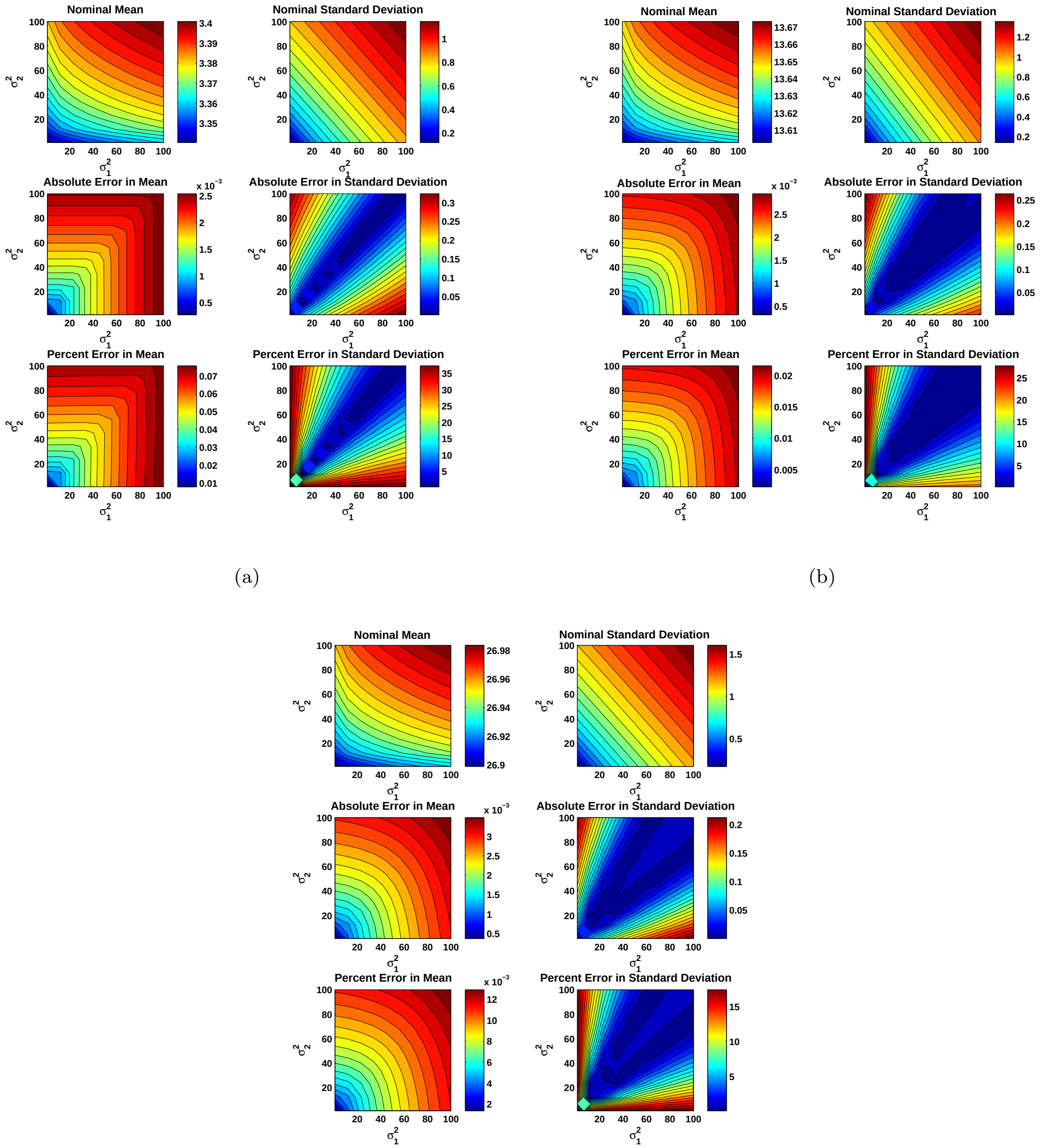
Table 5. Parameter ranges to assess the validity of the proposed methodology.

Parameter	Distribution
$\sigma^2_{u_{p,1}}$	$\mathcal{U}(0, 100)$
$\sigma^2_{u_{p,1}}$	$\mathcal{U}(0, 100)$
$\rho_{u_{p,1}u_{p,2}}$	$\mathcal{U}(-1, 1)$
$\mathbf{A_1'}$	$\begin{pmatrix} \mathcal{U}(-1,1) & \mathcal{U}(-1,1) \\ \mathcal{U}(-1,1) & \mathcal{U}(-1,1) \end{pmatrix}$
$\mathbf{A_2'}$	$\begin{pmatrix} \mathcal{U}(-1,1) & \mathcal{U}(-1,1) \\ \mathcal{U}(-1,1) & \mathcal{U}(-1,1) \end{pmatrix}$
$\mathbf{C_1'}$	$\begin{pmatrix} \mathcal{U}(-1,1) & \mathcal{U}(-1,1) \\ \mathcal{U}(-1,1) & \mathcal{U}(-1,1) \end{pmatrix}$
$\mathbf{B_1'}$	$\begin{pmatrix} \mathcal{U}(-1,1) \\ \mathcal{U}(-1,1) \end{pmatrix}$
$\mathbf{B_2'}$	$\begin{pmatrix} \mathcal{U}(-1,1) \\ \mathcal{U}(-1,1) \end{pmatrix}$

In addition to the parameters shown in Table 5, the effect of the mean of the probabilistic parameters was conducted by analyzing three different cases–one where the mean was $\boldsymbol{\mu}_{\mathbf{u_p}} = (0\ 0)^T$, one where the mean was $\boldsymbol{\mu}_{\mathbf{u_p}} = (100\ 0)^T$, and one where the mean was $\boldsymbol{\mu}_{\mathbf{u_p}} = (100\ 100)^T$. The results were then compared with results propagated analytically resulting in Fig. 8.

It is observed from these results that the mean error is less than 0.08% for all of the cases examined. This is a result of the system being linear and the Kalman filter propagating results exactly for a linear system. Therefore the error in the mean is solely a result of the convergence criterion being utilized. For each case, there is seen to be a rise in the standard deviation error near the origin. This is because the nominal mean goes to zero causing a rise in the in the percent error near this point.

The assessment methodology is observed to provide a consistent conservative bound on the variance as seen in Fig. 8 since all of the percent error values are positive. It is also interesting to note that the error in mean and standard deviation, regardless of the mean of the input, appears to be close to the same order of magnitude. As the mean input value increases, the magnitude of the mean response and standard deviation of that response increases, which causes a decrease in the percent error. Furthermore, it is observed that the maximum error approaches a limit of less than 40%. This limit is believed to be a function of the 2-norm being used. In future work, this limit will be explored and related to the dimensionality of the problem. In analyzing the data, the largest errors are caused for weakly coupled systems, that is systems where $\mathbf{C_1'}$ is small. This can be explained since $\mathbf{C_1'}$ being small leads to a larger domain of values that lead to a converged design. Additionally, since the interplay between $\mathbf{y_1}$ and $\mathbf{y_2}$ is reduced, the iterations to achieve convergence is reduced in these cases.



(a) (b)

(c)

Figure 8. Maximum error for a two contributing analysis multidisciplinary design with (a) $\boldsymbol{\mu}_{\mathbf{up}} = (0\ 0)^T$, (b) $\boldsymbol{\mu}_{\mathbf{up}} = (100\ 0)^T$, and (c) $\boldsymbol{\mu}_{\mathbf{up}} = (100\ 100)^T$.

IV.D. Comparison with Other Methods

The errors associated with the rapid linear robustness technique are compared to more commonly used methods to propagate uncertainty, namely a 10,000 case Monte Carlo analysis, the unscented transform, and fast probability integration. For the data presented, specific numerical values were utilized for the various problem matrices and vectors, these are given by

$$\mathbf{A}'_1 = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.5 \end{pmatrix} \quad \mathbf{B}'_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathbf{C}'_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{A}_2' = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.25 \end{pmatrix} \quad \mathbf{B}_2' = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The advantages of the rapid robust analysis technique are elucidated in the Table 6 where it can be seen that the proposed method provides similar levels of accuracy to the other contemporary methods. However, this level of accuracy is obtained for less than half the number of function evaluations compared to the unscented transform and orders of magnitude fewer function evaluations compared to Monte Carlo and fast probability integration. For known functions, this may not be a large advantage, but if the model for each CA needs to be built real-time, this provides a large execution time benefit. It should also be noted that in comparing the data to the exact propagation in the previous section, the proposed method captures the mean more accurately than any of the other techniques presented here (maximum error of 0.08% compared compared to relative errors that are an order of magnitude greater).

Table 6. Comparison of the performance of the rapid robustness assessment method with other multidisciplinary uncertainty assessment techniques.

	Monte Carlo	**Unscented Transform**	**Fast Probability Integration**
Maximum Mean Discrepancy (%)	0.48234	0.50458	0.50357
Maximum Standard Deviation Discrepancy (%)	2.3458	1.6789	2.3447
Increase in Number of CA Evaluations (-)	109,954 (78,539%)	435 (124%)	2,349 (1,678%)

V. Robust Design of a Two Bar Truss

Consider the planar truss which consists of two elements with a vertical load at the mutual joint, as shown in Fig. 9 (adapted from Ref. 64).

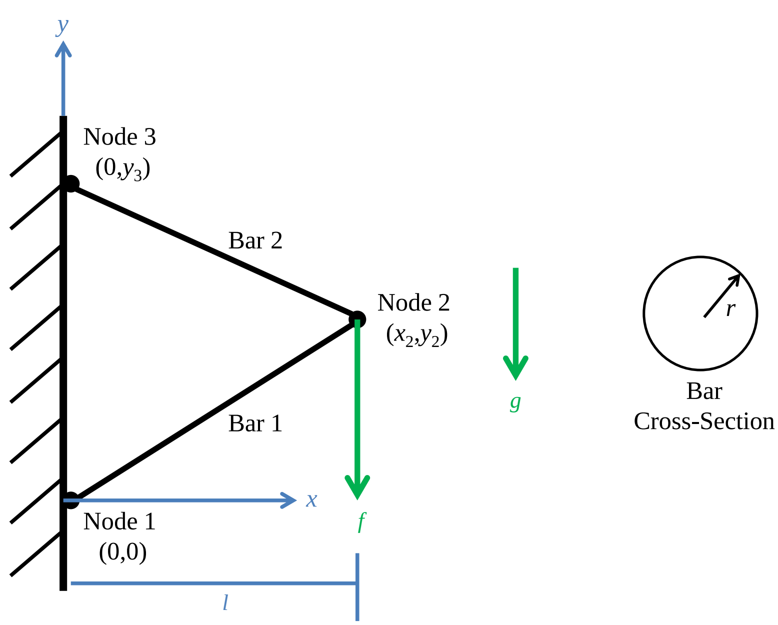


Figure 9. Two bar truss with a load at the mutual joint.

For this problem, it is desired to find the vertical position of nodes 2 and 3, y_2 and y_3, that minimize the mean weight of the truss while ensuring that the structure will not fail do to buckling or yielding with some factor of safety given fixed values for the material properties, E, σ_y, and ρ, the load, f, and the bar geometry, r_1 and r_2. The horizontal position of node 2 is constrained to be l. In standard form, the deterministic problem is written as

$$\left.\begin{array}{ll}
\textbf{Minimize}: & \mathcal{J} = \rho\pi\left(L_1 + L_2\right) = \rho\pi\left(\sqrt{l^2 + y_2^2} + \sqrt{l^2 + (y_3 - y_2)^2}\right) \\
\textbf{Subject to}: & g_1(y_2, y_3) = |T_1(y_2, y_3)| - \pi r_1^2 \sigma_y \leq 0 \\
 & g_2(y_2, y_3) = |T_2(y_2, y_3)| - \pi r_2^2 \sigma_y \leq 0 \\
 & g_3(y_2, y_3) = -T_1(y_2, y_3) - \dfrac{\pi^2 E I_1}{L_1^2} \leq 0 \\
 & g_4(y_2, y_3) = -T_2(y_2, y_3) - \dfrac{\pi^2 E I_2}{L_2^2} \leq 0 \\
\textbf{By varying}: & y_2, y_3
\end{array}\right\}$$

where L_1 and L_2 are the lengths of the two bars, respectively, I_1 and I_2 are the moments of inertia of the two bars ($I_i = \frac{1}{4} m r_i^2$), and $T_1(y_2, y_3)$ and $T_2(y_2, y_3)$ are the tensions in the two bars. Numerical values for this problem are shown in Table 7

Table 7. Parameters for the two-bar truss problem.

Parameter	Description	Nominal Value	Distribution
E	Young's Modulus	200×10^6 kN/m^2	–
σ_y	Yield Strength	250×10^3 kN/m^2	$\mathcal{N}(250 \times 10^3, 625 \times 10^6)$
ρ	Density	7850 kg/m^3	$\mathcal{N}(7850, 100)$
l	Length	5 m	–
r_1	Radius of Bar 1	30 mm	–
r_2	Radius of Bar 2	5 mm	–
f	Applied Force	3.5 kN	$\mathcal{N}(3.5, 0.49)$
g	Gravitational Acceleration	9.81 m/s^2	–

Step 1: Decompose the Design

Two analyses must occur in order to design the two bar truss: a structural analysis and a sizing of the bars constituting the truss. Although not explicit in the problem statement, the mass of the bars also provide a load through their weight. Hence, this is a coupled analysis problem since the structural analysis depends on the sizing of each of the bars. The coupled DSM is shown in Fig. 10.

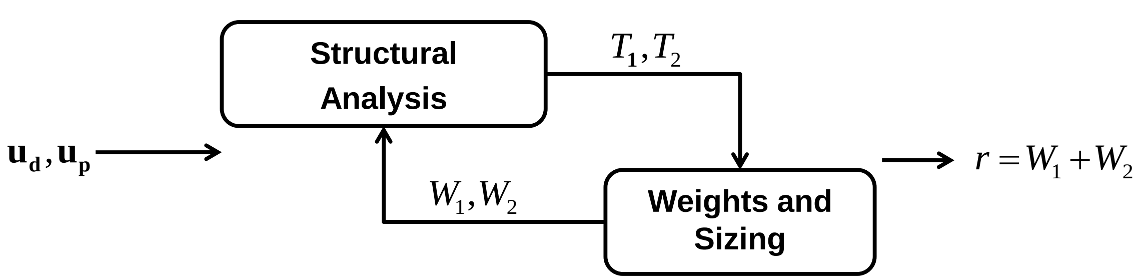


Figure 10. Two bar truss design structure matrix.

The inputs into the design problem are the deterministic and probabilistic parameters of the problem whose values are shown in Table 7. In particular,

$$\mathbf{u_d} = \begin{pmatrix} E & l & r_1 & r_2 & g & y_2 & y_3 \end{pmatrix}^T$$

and

$$\mathbf{u_p} = \begin{pmatrix} \sigma_y & \rho & f \end{pmatrix}^T$$

The structural analysis CA feeds the forces seen in each of the members of the truss to the weights and sizing module. These can be found through the static equilibrium equations and are found by solving the linear equations

$$\begin{pmatrix} \frac{l}{L_1} & 0 & 1 & 0 & 0 & 0 \\ -\frac{y_2}{L_1} & 0 & 0 & 1 & 0 & 0 \\ -\frac{l}{L_1} & \frac{l}{L_2} & 0 & 0 & 0 & 0 \\ -\frac{y_2}{L_1} & \frac{y_3 - y_2}{L_2} & 0 & 0 & 0 & 0 \\ 0 & \frac{l}{L_2} & 0 & 0 & 1 & 0 \\ 0 & \frac{y_3 - y_2}{L_2} & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ R_{1x} \\ R_{1y} \\ R_{3x} \\ R_{3y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{l}{2y_3}(W_1 + W_2 + 2f) \\ f\left(1 + \frac{y_2}{y_3}\right) \\ 0 \\ 0 \\ f \\ \frac{l}{2y_3}(W_1 + W_2 + 2f) \\ f\left(1 + \frac{y_2}{y_3}\right) + W_1 + W_2 \end{pmatrix}$$

for the tensions. The weights and sizing CA computes the weights of each of the bars based on the relationship

$$\mathbf{y_2} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} \pi \rho g r_1^2 L_1 \\ \pi \rho g r_2^2 L_2 \end{pmatrix}$$

Both relationships defined by the CAs rely on the lengths of the bars, which are given by

$$\begin{aligned} L_1 &= \sqrt{l^2 + y_2^2} \\ L_2 &= \sqrt{l^2 + (y_3 - y_2)^2} \end{aligned}$$

This is a realistic example in which the CAs are nonlinear. Therefore, in order to apply the developed methodology, a Taylor series expansion about a nominal value (chosen to be the previous iterate or mean value) must be conducted. Functionally, this means that the expression for the tensions, $\mathbf{y_1}$, can be expanded as follows

$$\mathbf{y_1} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \approx \begin{pmatrix} \left.\frac{\partial T_1}{\partial \mathbf{u_d}}\right|_{\hat{\mathbf{u}}_\mathbf{d}} (\mathbf{u_d} - \hat{\mathbf{u}}_\mathbf{d}) + \left.\frac{\partial T_1}{\partial \mathbf{u_p}}\right|_{\boldsymbol{\mu}_{\mathbf{u_p}}} (\mathbf{u_p} - \boldsymbol{\mu}_{\mathbf{u_p}}) + \left.\frac{\partial T_1}{\partial \mathbf{y}}\right|_{\hat{\mathbf{y}}} (\mathbf{y} - \hat{\mathbf{y}}) \\ \left.\frac{\partial T_2}{\partial \mathbf{u_d}}\right|_{\hat{\mathbf{u}}_\mathbf{d}} (\mathbf{u_d} - \hat{\mathbf{u}}_\mathbf{d}) + \left.\frac{\partial T_2}{\partial \mathbf{u_p}}\right|_{\boldsymbol{\mu}_{\mathbf{u_p}}} (\mathbf{u_p} - \boldsymbol{\mu}_{\mathbf{u_p}}) + \left.\frac{\partial T_2}{\partial \mathbf{y}}\right|_{\hat{\mathbf{y}}} (\mathbf{y} - \hat{\mathbf{y}}) \end{pmatrix}$$

Similarly, the expression for $\mathbf{y_2}$ can be expanded as

$$\mathbf{y_2} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \approx \begin{pmatrix} \pi \rho g r_1^2 \frac{\hat{y}_2}{\sqrt{l^2 + \hat{y}_2^2}} (y_2 - \hat{y}_2) \\ \pi \rho g r_2^2 \left(\frac{\hat{y}_2 - \hat{y}_3}{\sqrt{l^2 + (\hat{y}_2 - \hat{y}_3)^2}} (y_2 - \hat{y}_2) + \frac{\hat{y}_3 - \hat{y}_2}{\sqrt{l^2 + (\hat{y}_2 - \hat{y}_3)^2}} (y_3 - \hat{y}_3) \right) \end{pmatrix}$$

Therefore, in the form of Eq. (30)

$$\mathbf{A_1} = \left(\left.\frac{\partial T_1}{\partial \mathbf{y}}\right|_{\hat{\mathbf{y}}} \right) \quad \mathbf{B_1} = \left(\left.\frac{\partial T_1}{\partial \mathbf{u_d}}\right|_{\hat{\mathbf{u}}_\mathbf{d}} \right) \quad \mathbf{C_1} = \left(\left.\frac{\partial T_1}{\partial \mathbf{u_p}}\right|_{\boldsymbol{\mu}_{\mathbf{u_p}}} \right)$$

$$\mathbf{d_1} = -\left(\left.\frac{\partial T_1}{\partial \mathbf{u_d}}\right|_{\hat{\mathbf{u}}_\mathbf{d}} \hat{\mathbf{u}}_\mathbf{d} + \left.\frac{\partial T_1}{\partial \mathbf{u_p}}\right|_{\boldsymbol{\mu}_{\mathbf{u_p}}} \boldsymbol{\mu}_{\mathbf{u_p}} + \left.\frac{\partial T_1}{\partial \mathbf{y}}\right|_{\hat{\mathbf{y}}} \hat{\mathbf{y}} \right)$$

$$\mathbf{A_2} = \left(\left.\frac{\partial T_2}{\partial \mathbf{y}}\right|_{\hat{\mathbf{y}}} \right) \quad \mathbf{B_2} = \left(\left.\frac{\partial T_2}{\partial \mathbf{u_d}}\right|_{\hat{\mathbf{u}}_\mathbf{d}} \right) \quad \mathbf{C_2} = \left(\left.\frac{\partial T_2}{\partial \mathbf{u_p}}\right|_{\boldsymbol{\mu}_{\mathbf{u_p}}} \right)$$

$$\mathbf{d_2} = -\left(\left.\frac{\partial T_2}{\partial \mathbf{u_d}}\right|_{\hat{\mathbf{u}}_\mathbf{d}} \hat{\mathbf{u}}_\mathbf{d} + \left.\frac{\partial T_2}{\partial \mathbf{u_p}}\right|_{\boldsymbol{\mu}_{\mathbf{u_p}}} \boldsymbol{\mu}_{\mathbf{u_p}} + \left.\frac{\partial T_2}{\partial \mathbf{y}}\right|_{\hat{\mathbf{y}}} \hat{\mathbf{y}} \right)$$

Step 2: Identify the Random Variables and their Distributions

All of the random variables in this example are associated with the parameters and not with the model itself. As shown in Table 7, the values are given by $\sigma_y \sim \mathcal{N}(250 \times 10^3, 625 \times 10^6)$, $\rho \sim \mathcal{N}(7850, 100)$, and $f \sim \mathcal{N}(3.5, 0.49)$.

Step 3: Form the Iterative Equations

In order to use the Kalman filter to simultaneously estimate robustness and converge the design, the iterative equations described in Eq. (31) for fixed-point iteration need to be formed. Through analogy of variables, the matrices are given by

$$\boldsymbol{\Lambda} = \begin{pmatrix} \mathbf{A_1} \\ \mathbf{A_2} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \left.\frac{\partial T_1}{\partial \mathbf{y_2}}\right|_{\hat{\mathbf{y}}} \\ \left.\frac{\partial T_2}{\partial \mathbf{y_1}}\right|_{\hat{\mathbf{y}}} & \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{\beta} = \begin{pmatrix} \mathbf{B_1} \\ \mathbf{B_2} \end{pmatrix} = \begin{pmatrix} \left.\frac{\partial T_1}{\partial \mathbf{u_d}}\right|_{\hat{\mathbf{u}}_\mathbf{d}} \\ \left.\frac{\partial T_2}{\partial \mathbf{u_d}}\right|_{\hat{\mathbf{u}}_\mathbf{d}} \end{pmatrix}$$

$$\boldsymbol{\gamma} = \begin{pmatrix} \mathbf{C_1} \\ \mathbf{C_2} \end{pmatrix} = \begin{pmatrix} \left.\frac{\partial T_1}{\partial \mathbf{u_p}}\right|_{\boldsymbol{\mu}_{\mathbf{u_p}}} \\ \left.\frac{\partial T_2}{\partial \mathbf{u_p}}\right|_{\boldsymbol{\mu}_{\mathbf{u_p}}} \end{pmatrix}$$

$$\boldsymbol{\delta} = \begin{pmatrix} \mathbf{d_1} \\ \mathbf{d_2} \end{pmatrix} = \begin{pmatrix} -\left(\left.\frac{\partial T_1}{\partial \mathbf{u_d}}\right|_{\hat{\mathbf{u}}_\mathbf{d}} \hat{\mathbf{u}}_\mathbf{d} + \left.\frac{\partial T_1}{\partial \mathbf{u_p}}\right|_{\boldsymbol{\mu}_{\mathbf{u_p}}} \boldsymbol{\mu}_{\mathbf{u_p}} + \left.\frac{\partial T_1}{\partial \mathbf{y}}\right|_{\hat{\mathbf{y}}} \hat{\mathbf{y}} \right) \\ -\left(\left.\frac{\partial T_2}{\partial \mathbf{u_d}}\right|_{\hat{\mathbf{u}}_\mathbf{d}} \hat{\mathbf{u}}_\mathbf{d} + \left.\frac{\partial T_2}{\partial \mathbf{u_p}}\right|_{\boldsymbol{\mu}_{\mathbf{u_p}}} \boldsymbol{\mu}_{\mathbf{u_p}} + \left.\frac{\partial T_2}{\partial \mathbf{y}}\right|_{\hat{\mathbf{y}}} \hat{\mathbf{y}} \right) \end{pmatrix}$$

where the numerical values for each of these matrices is evaluated at each subsequent iteration at the appropriate nominal values.

Step 4: Ensure a Solution Exists

While this problem is posed as a linear system, the matrix $\boldsymbol{\Lambda}$ varies with iteration. This requires a Lyapunov analysis to be conducted in order to identify the stability of the system. For this example, this analysis was completed simultaneously with the convergence by numerically solving a matrix Riccati equation. A positive definite matrix, $\mathbf{P}$, for the quadratic problem was able to be found that satisfies the relationship

$$\boldsymbol{\Lambda}_k^T \mathbf{P}_k \boldsymbol{\Lambda}_k - \mathbf{P}_k = \mathbf{Q}_k$$

for $\mathbf{Q}_k > 0$. Since a solution for $\mathbf{P}_k$ was able to be found when $\mathbf{Q}_k = \mathbf{I}_{\mathbf{4}\times\mathbf{4}}$ for each iterate, this enables a Lyapunov function of the form

$$V_k = \mathbf{y}_k^T \mathbf{P}_k \mathbf{y}_k$$

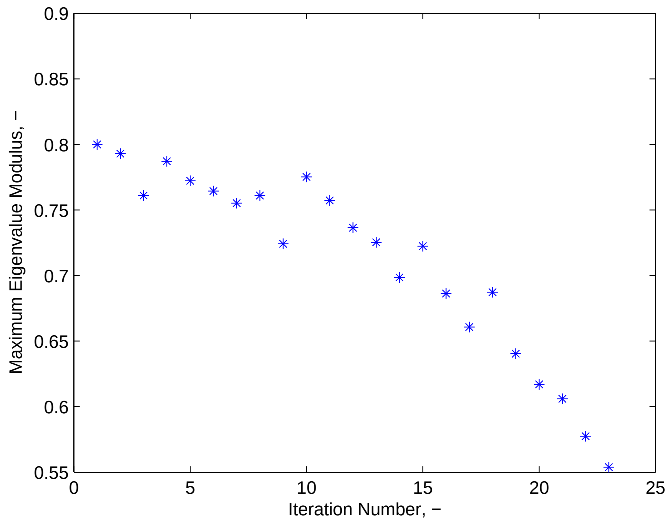


Figure 11. History of the modulus of the maximum eigenvalue of the two bar truss system with iteration.

to be formed which shows asymptotic stability for a linear, time varying, discrete system. In addition to using Lyapunov stability, the modulus of the eigenvalues show similar conclusions regarding the asymptotic stability as shown in Fig. 11.

Step 5: Estimate the Mean Output and the Covariance

The equations formed in the prior step can then be propagated through the Kalman filter defined by Eqs. (20)-(26) with

$$\mathbf{F}_{k-1} = \mathbf{\Lambda}, \quad \forall k \in \{1, 2, ..., n\}$$

$$\mathbf{B}_{k-1} = \begin{pmatrix} \boldsymbol{\beta} & \boldsymbol{\gamma} & \mathbf{I_{4\times4}} \end{pmatrix}, \quad \forall k \in \{1, 2, ..., n\}$$

$$\mathbf{u}_{k-1} = \begin{pmatrix} \mathbf{u_d} \\ \mathbf{u_p} \\ \boldsymbol{\delta} \end{pmatrix}, \quad \forall k \in \{1, 2, ..., n\}$$

where in this example $\mathbf{u_d} = \begin{pmatrix} E & l & r_1 & r_2 & g & y_2 & y_3 \end{pmatrix}^T$ and $\mathbf{u_p} = \mathbb{E}(\mathbf{u_p}) = \boldsymbol{\mu}_{\mathbf{u_p}}$. In this example, the matrix $\mathbf{Q}$ is the null matrix since the only uncertain parameters of the problem are associated with the input parameters, not the model. The unscented transform is used on an uncoupled system with the distribution described in Step 2 in order to identify $\mathbf{y_0}$ and $\mathbf{\Sigma_0}$, the initial output mean and covariance for each design. A design is considered converged when the absolute difference between iteration estimates is less than 1×10^{-4} or the relative difference is less than 1×10^{-6}.

Step 6: Identify the Mean and Variance Bound of the Objective Function

Upon convergence, the value of $\mathbf{y}$, the state variable in the problem, is the mean response for each of the components of the output CAs. In this example, the matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix}$$

since the objective is the weight of truss $W_1 + W_2$, the two elements of the second CA output. Therefore,

$$\bar{r} \approx \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \hat{\mathbf{y}}_{n|n}$$

The estimate for the variance (*i.e.*, the variance bound) in this case is two times the 2-norm of the entire estimated covariance matrix

$$\sigma_r^2 \approx \sum_{i=1}^{2} \parallel \mathbf{\Sigma}_{\mathbf{y}^*{}_{n|n}} \parallel_2 = 2 \parallel \mathbf{\Sigma}_{\mathbf{y}^*{}_{n|n}} \parallel_2$$

.

Step 7: Optimize for Uncertainty and Ensure Constraints are Met
Formulating the output of Step 6 in terms of the mean and variance allows for an optimal control problem to be setup for the system's design, where the objective function is defined by

$$\mathcal{J} = \alpha \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \hat{\mathbf{y}}_{n|n} + 2\beta \parallel \mathbf{\Sigma}_{\boldsymbol{y}^*}{}_{n|n} \parallel_2$$

and α and β allow different weighting on the mean and variance. The constraints for this problem as given are

$$\mathbf{g}(\mathbf{y}, \mathbf{u}) = \begin{pmatrix} g_1(\mathbf{y}, \mathbf{u}) \\ g_2(\mathbf{y}, \mathbf{u}) \\ g_3(\mathbf{y}, \mathbf{u}) \\ g_4(\mathbf{y}, \mathbf{u}) \end{pmatrix} = \begin{pmatrix} |T_1(\mathbf{u})| - \pi r_1^2 \sigma_y \\ |T_2(\mathbf{u})| - \pi r_2^2 \sigma_y \\ -T_1(\mathbf{u}) - \dfrac{\pi^2 E I_1}{L_1^2} \\ -T_2(\mathbf{u}) - \dfrac{\pi^2 E I_2}{L_2^2} \end{pmatrix}$$

Additionally, there is an equality constraint for the control that states

$$\mathbf{h}(\mathbf{y}, \mathbf{u}) = \mathbf{u}_k - \mathbf{u}_{k-1} = \mathbf{0}, \quad \forall k \in \{1, \ldots, n\}$$

Therefore, the Hamiltonian in the optimal control problem is given by

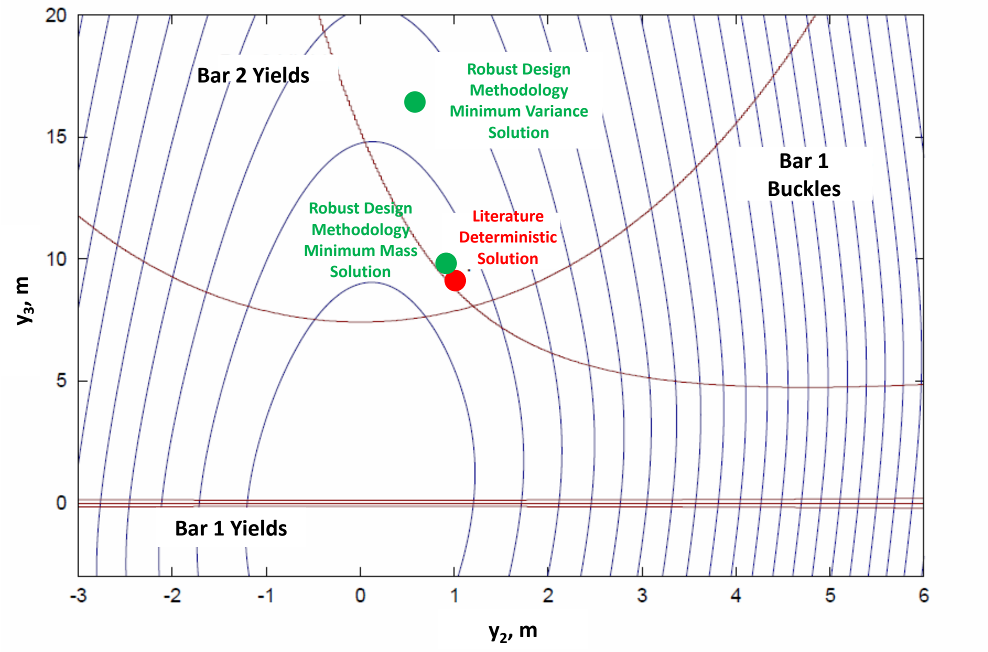
$$H(\mathbf{y}, \mathbf{u}) = \psi_0 \left(\alpha \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \hat{\mathbf{y}}_{n|n} + 2\beta \parallel \mathbf{\Sigma}_{\boldsymbol{y}^*}{}_{n|n} \parallel_2 \right) + \boldsymbol{\gamma}^T \left(\mathbf{u}_k - \mathbf{u}_{k-1} \right) + \boldsymbol{\lambda}^T \begin{pmatrix} |T_1(\mathbf{u})| - \pi r_1^2 \sigma_y \\ |T_2(\mathbf{u})| - \pi r_2^2 \sigma_y \\ -T_1(\mathbf{u}) - \dfrac{\pi^2 E I_1}{L_1^2} \\ -T_2(\mathbf{u}) - \dfrac{\pi^2 E I_2}{L_2^2} \end{pmatrix}$$

where the terms in this relation can be computed numerically. The optimal control conditions can then be used to compute the values of y_2 and y_3 for a chosen value of α and β such as described in Ref. 13.

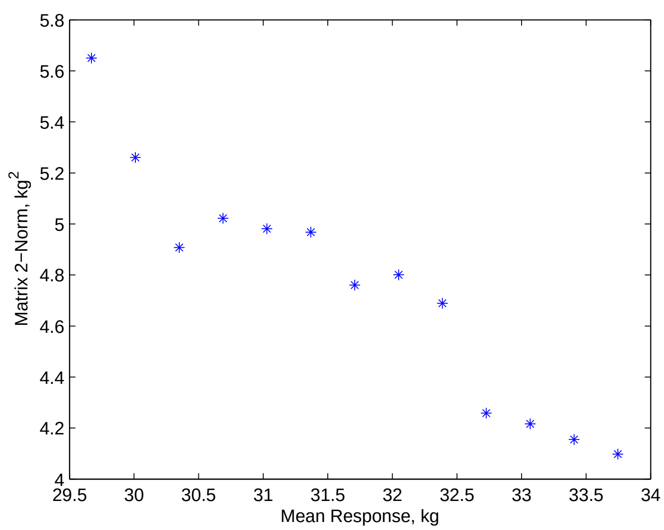
V.A. Design Results

As mentioned at the beginning of this section, the formulation of this problem is based on work by 64. This work showed the deterministic optimal design to be as shown in Fig. 12(a) which also shows the minimum variance robust design found in this work.[64] This positions the vertical positions of the nodes, (y_2^*, y_3^*), at $(0.751, 9.970)$ m with an objective function value of $\mathcal{J} = 29.673$ kg. The deterministic case of this analysis (*i.e.*, when $\alpha = 1$ and $\beta = 0$) yields a very similar result with (y_2^*, y_3^*) at $(0.746, 9.991)$ m with an objective function value of $\mathcal{J} = 29.679$ kg which implies that the method developed achieves an accurate numerical result even in the case of nonlinear problems.

The variation in terms of mean and variance for this problem is shown in Fig. 12(b). From this figure, the deterministic optimum is the minimum mean solution; however, it is not the minimum variance solution. This minimum variance design is approximately 4 kg heavier.



(a)



(b)

Figure 12. Design solutions for the two-bar truss showing the (a) design space with the optimal designs and (b) variation of $2||\mathbf{\Sigma}_{\mathbf{y}^*}||_2$ with the mean objective function.

VI. Conclusions

In this work, the foundations has been described for how to cast the general multidisciplinary design problem as a dynamical system. This is a process of identifying the iteration scheme and then formulating the CAs to fit this iteration scheme for both steps of the design process: identifying candidate designs and finding the optimum of these candidate designs. Three techniques from dynamical system theory were described with particular emphasis being placed on their role in the formation of a rapid robust design methodology for multidisciplinary design. These were (1) stability analysis to examine the existence of a converged design (for a given iteration scheme), (2) optimal control to handle equality and inequality constraints by obtaining an additional set of constraints and adjoining these to the desired objective function, and (3) estimation theory to obtain design robustness characteristics. These techniques were demonstrated together to provide a conservative upper bound of the variance of the design to a scalar objective function.

A probabilistic performance assessment of the bounding robustness methodology as well as an example robust design problem were provided, which showed aspects of dynamical system theory being applied to multidisciplinary design. The probabilistic assessment showed that robustness assessment methodology had maximum errors relative to exact values less than 1% on the mean objective value and less than 40% for the standard deviation for a large design space. In adddition, for specific values of the CAs a comparison between traditional uncertainty quantification techniques and the robustness assessment methodology was provided. This example demonstrated significant computational performance gains of the proposed technique with a reduction in the number of function calls by a factor of two when compared to the unscented tranform and more than an order of magnitude when compared to fast probability integration and Monte Carlo techniques with minimal sacrifices to the accuracy ($< 3\%$). In the design example, the methodology was applied to a nonlinear problem, that of design a two-bar truss system. The successive linearization procedure showed a minimal mass design that performs better than that in the literature and characterized the difference between optimal and robust design.

References

[1] Appa, K. and Argyris, J., "Non-linear multidisciplinary design optimization using system identification and optimal control theory," *Computer Methods in Applied Mechanics and Engineering*, Vol. 128, 1995.

[2] Smith, R. P., Eppinger, S. D., and Gopal, A., "Testing an Engineering Design Iteration Model in an Experimental Setting," Tech. Rep. WP 3386-92-MS, Massachusetts Institute of Technology, Cambridge, Massachusetts, Feb. 1992.

[3] Lewis, K. and Mistree, F., "Modeling Interactions in Multisciplinary Design: A Game Theoretic Approach," *AIAA Journal*, Vol. 35, No. 8, 1998.

[4] DeLaurentis, D. A., *A Probabilistic Approach to Aircraft Design Emphasizing Stability and Control Uncertainties*, Ph.D. thesis, Georgia Institute of Technology, Atlanta, GA, Nov. 1998.

[5] Grant, M. J., Clark, I. G., and Braun, R. D., "Rapid Entry Corridor Trajectory Optimization for Conceptual Design," *AIAA 2010-7810*, Toronto, Ontario, Canada, Aug. 2010.

[6] Grant, M. J., *Rapid Simultaneous Hypersonic Aerodynamic and Trajectory Optimization for Conceptual Design*, Ph.D. thesis, Georgia Institute of Technology, Georgia, May 2012.

[7] Brogan, W. L., *Modern Control Theory*, Prentice Hall, Upper Saddle River, NJ, 1991.

[8] Scheinverman, E. R., *Invitation to Dynamical Systems*, Prentice Hall, Upper Saddle River, NJ, 1995.

[9] Vanderplaats, G. N., *Numerical Optimization Techniques for Engineering Design*, Vanderplaats Research & Development, Inc., 2005.

[10] Khalil, H. K., *Nonlinear Systems, 3rd Ed.*, Prentice Hall, 2002.

[11] Kalman, R. E. and Bertram, J. E., "Control System Analysis and Design Via the "Second Method" of Lyapunov: I—Continuous-Time Systems," *Journal of Basic Engineering*, Vol. 82, No. 2, 1960.

[12] Kalman, R. E. and Bertram, J. E., "Control System Analysis and Design Via the "Second Method" of Lyapunov: II—Discrete-Time Systems," *Journal of Basic Engineering*, Vol. 82, No. 2, 1960.

[13] Boltyanskii, V. G., *Optimal Control of Discrete Systems*, John Wiley & Sons, Inc., New York, NY, 1978.

[14] Ogata, K., *Discrete-Time Control Systems*, Prentice Hall, Upper Saddle River, NJ, 1995.

[15] Bryson, A. E. and Ho, Y.-C., *Applied Optimal Control*, Taylor & Francis, New York, NY, 1975.

[16] Hull, D. G., *Optimal Control Theory for Applications*, Springer-Verlag, New York, NY, 2003.

[17] Hogg, R. V. and Tanis, E. A., *Probability and Statistical Inference*, Pearson Prentice Hall, Upper Saddle River, NJ, 2006.

[18] Roos, O. V., "A Formal Solution of Liouville's Equation," Tech. Rep. 32-10, Jet Propulsion Laboratory, March 1960.

[19] Fuller, A. T., "Analysis of Nonlinear Stochastic Systems by Means of the Fokker-Planck Equation," *International Journal of Control*, Vol. 9, No. 6, 1969.

[20] Risken, H., *The Fokker-Planck Equation: Methods of Solution and Applications*, Springer, New York, NY, 1996.

[21] Kalos, M. H. and Whitlock, P. A., *Monte Carlo Methods*, Wiley-VCH, 2008.

[22] Chen, W., Allen, J. K., Schrage, D. P., and Mistree, F., "Statistical Experimentation Methods for Achieving Affordable Concurrent Systems Design," *AIAA Journal*, Vol. 35, No. 5, 1997.

[23] Unal, R., Stanley, D., Engelund, W., and Lepsch, R., "Design for Quality Using Response Surface Methods: An Alternative to Taguchi's Parameter Design Approach," *Engineering Management Journal*, Vol. 6, No. 3, 1994.

[24] Klenijen, J. P., *Statistical Tools for Simulation Practitioners*, Marcel Dekker, New York, NY, 1987.

[25] Chen, W., Tsui, K.-L., Allen, J. K., and Mistree, F., "Integration of Response Surface Method with the Compromise Decision Support Problem in Developing a General Robust Design Procedure," *Advances in Design Automation*, Vol. 82, No. 2, 1995.

[26] Mavris, D. N. and Bandte, O., "Economic Uncertainty Assessment Using a Combined Design of Experiments/Monte Carlo Simulation Approach with Application to an HSCT," *17th Annual Conference of the International Society of Parametric Analysts*, San Diego, CA, May 1995.

[27] Madsen, H. O., Krenk, S., and Lind, N. C., *Methods of Structural Safety*, Prentice Hall, Englewood Cliffs, NJ, 1986.

[28]Wu, Y. T., "Structural Reliability Analysis Methods for Implicit Performance Functions," *Proceedings of the ASCE Special Conference in Probabilistic Mechanics and Structural and Geotechnical Reliability*, New York, NY, 1992.

[29]Khalessi, M. R., Wu, Y. T., and Torng, T. Y., "Most-Probable-Point-Locus Reliability Method in Standard Normal Space," *ASME Design Engineering*, Vol. 30, 1991.

[30]Wirshing, P. H. and Martin, W. S., "Fracture Mechanics Fatigue Reliability Assessment Employing the Most Probable Point Locus Method," *Proceedings of the the 5th ASCE Special Conference in Probabilistic Methods in Civil Engineering*, New York, NY, 1988.

[31]Wu, Y. T., Millwater, H. R., and Cruse, T. A., "An Advanced Probabilistic Structural Analysis Method for Implicit Performance Functions," *AIAA Journal*, Vol. 28, No. 9, 1990.

[32]Liu, W. K. and Belytschko, T., *Computational Mechanics of Reliability Analysis*, Elmepress International, Lauseanne, WA, 1989.

[33]Wu, Y. T. and Wirsching, P. H., "Demonstration of a Fast, New Probability Integration Method for Reliability Analysis," *ASME Journal of Engineering for Industry*, Vol. 109, 1987.

[34]Wu, Y. T. and Wirsching, P. H., "Advanced Reliability Methods for Probabilistic Structural Analysis," *ASCE Structural Safety and Reliability*, Vol. 3, 1989.

[35]Southwest Reserach Institute, San Antonio, TX, *FPI User's Manual and Theoretical Manual*, 1995.

[36]Du, X. and Chen, W., "A Most Probable Point Based Method for Uncertainty Analysis," *Journal of Design and Manufacturing Automation*, Vol. 4, 2001.

[37]Mahadevan, S. and Cruse, T. A., "Advanced First-Order Method for System Reliability," *Proceedings of the the ASCE Special Conference in Probabilistic Mechanics and Geotechnical Reliability*, New York, NY, 1992.

[38]Lin, H. Z. and Khalessi, M. R., "Calculation of Failure Probability by Using X-Space Most-Probable-Point," *AIAA 93-1624*, La Jolla, CA, 1993.

[39]Geller, D. K., "Linear Covariance Techniques for Orbial Rendezvous Analysis and Autonomous Onboard Mission Planning," *Journal of Guidance, Control, and Dynamics*, Vol. 29, No. 6, 2006, pp. 1404–1414.

[40]Christensen, D. and Geller, D., "Terrain-Relative and Beacon-Relative Navigation for Lunar Powered Descent and Landing," *AAS 009-057*, Breckenridge, CO, Jan. 2009.

[41]Geller, D. K. and Christensen, D. P., "Linear Covariance Analysis for Powered Lunar Descent and Landing," *Journal of Spacecraft and Rockets*, Vol. 46, No. 6, 2009.

[42]Du, X. and Chen, W., "Methodolgoy for Managing the Effect of Uncertainty in Simulation-Based Design," *AIAA Journal*, Vol. 38, No. 8, 2000.

[43]Kawata, T., "Fourier Analysis in Probability Theory," *Probability and Mathematical Statistics, A Series of Monographs and Textbooks*, Vol. 55, No. 3, 1975.

[44]Hoybye, J. A., "Model Error Propagation and Data Collection Design: An Application in Water Quality Modeling," *Water, Air, and Soil Pollution*, Vol. 103, No. 1, 1998.

[45]Isukapalli, S. S. and Georgopoulos, P. G., "Stochastic Response Surface Methods (SRSMs) for Uncertainty Propagation: Application to Environmental and Biological Systems," *Risk Analysis*, Vol. 18, No. 3, 1998.

[46]Cheng, W., *Parametric Uncertainty Analysis for Complex Engineering Systems*, Ph.D. thesis, Massachusetts Institute of Technology, Massachussetts, June 1999.

[47]Kumar, M., Singla, P., Chakravorty, S., and Junkins, J. L., "A Multi-Resolution Approacch for Steady State Uncertainty Determination in Nonlinear Dynamic Systems," *Proceedings of the 38th Southeastern Symposium on System Theory*, Cookeville, TN, March 2006.

[48]Kumar, M., Singla, P., Chakravorty, S., and Junkins, J. L., "The Partition of Unity Finite Element Approach to the Stationary Fokker-Planck Equation," *AIAA 2006-6285*, Keystone, CO, Aug. 2006.

[49]Muscolino, G., Ricciardi, G., and Vasta, M., "Stationary and Nonstationary Probability Density Function for Nonlinear Oscillators," *International Journal of Non-Linear Mechanics*, Vol. 32, No. 6, 1997.

[50]Paola, M. D. and Sofi, A., "Approximate Solution of the Fokker-Planck-Kolmogorov Equation," *Probabilistic Engineering Mechanics*, Vol. 17, No. 4, 2002.

[51]Iyengar, R. N. and Dash, P. K., "Study of the Rangom Vibration of Nonlinear Systems by the Gaussian Closure Technique," *Journal of Applied Mechanics*, Vol. 45, No. 1, 1978.

[52]Roberts, J. B. and Spanos, P. D., *Random Vibration and Statistical Linearization*, Wiley, New York, NY, 1990.

[53]Lefebvre, T., Bruyninckx, H., and Schutter, J. D., "Kalman Filters of Non-Linear Systems: A Comparison of Performance," *International Journal of Control*, Vol. 77, No. 7, 2004.

[54]Lefebvre, T., Bruyninckx, H., and Schutter, J. D., "Comment on a New Method for the Nonlinear Transformations of Means and Covariances," *IEEE Transactions on Automatic Control*, Vol. 47, No. 8, 2002.

[55]Terejanu, G., Singla, P., Singh, T., and Scott, P. D., "Uncertainty Propagation for Nonlinear Dynamic Systems Using Gaussian Mixture Models," *Journal of Guidance, Control, and Dynamics*, Vol. 31, No. 6, 2008.

[56]Kalman, R. E., "A New Approach to Linear Filtering and Prediction Problems," *Journal of Basic Engineering*, Vol. 82, 1961.

[57]Kalman, R. E. and Bucy, R. S., "New Results in Linear Filtering and Prediction Theory," *Journal of Basic Engineering*, Vol. 83, 1961.

[58]Brammer, K. and Siffling, G., *Kalman-Bucy Filters*, Artesch House, Norwood, MA, 1989.

[59]Chui, C. K. and Chen, G., *Kalman Filtering with Real-Time Applications*, Springer-Verlag, New York, NY, 1991.

[60]Simon, D., *Optimal State Estimation*, John Wiley & Sons, Hoboken, NJ, 2006.

[61]Gelb, A., *Applied Optimal Estimation*, The MIT Press, Cambridge, MA, 1971.

[62]Otero, R. E. and Braun, R. D., "The Planetary Entry Systems Synthesis Tool: A Conceptual Design and Analysis Tool for EDL Systems," *IEEEAC 1331*, Big Sky, MT, March 2010.

[63]Papachristodoulou, A. and Prajna, S., "On the Construction of Lyapunov Functions using the Sum of Squares Decomposition," *Proceedings of the 41st IEEE Conference on Decision and Control*, Piscatway, NJ, Dec. 2002.

[64]Gavin, H. P., "Introductory Optimization Example: Design of a Two-Bar Truss," Tech. rep., Duke University, 2012.