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Using Estimation Techniques in Multidisciplinary Design

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Viewing the multidisciplinary design problem as a dynamical system a number of tools from the established field of dynamical system theory became available to the multidisciplinary design community. This work demonstrates the applicability of applying the Kalman filter in a manner similar to linear covariance analysis to the multidisciplinary design problem to obtain robustness characteristics. In addition to robustness characteristics, the estimation theory is shown to be applicable to design decomposition. Following theoretical development, two example problems demonstrate the applicability of applying dynamical system theory. For a linear, two contributing analysis problem showed the mean was able to be estimated with an error less than 0.08% and a matrix norm bounded the variance to less than 37.8% relative to analytic propagation. This error is shown to be a function of the geometry of the matrix two-norm and reduces as the problem dimensionality increases. The use of estimation theory is also shown to be applicable for nonlinear designs through a two-bar truss problem through successive linearization.

Nomenclature

$(\cdot)_{j\|k}$	Estimate at j given observations up to and including k
$\boldsymbol{\beta}$	Deterministic input contribution in the fixed-point iteration equation, $\boldsymbol{\beta} \in \mathbb{R}^{m \times d}$
$\boldsymbol{\delta}$	Bias in the fixed-point iteration equation, $\boldsymbol{\delta} \in \mathbb{R}^{m}$
$\boldsymbol{\gamma}$	Probabilistic input contribution in the fixed-point iteration equation, $\boldsymbol{\gamma} \in \mathbb{R}^{m \times p}$
$\boldsymbol{\Lambda}$	State contribution in the fixed-point iteration equation, $\boldsymbol{\Lambda} \in \mathbb{R}^{m \times m}$
$\boldsymbol{\lambda}$	Lagrange multiplier
$\boldsymbol{\Sigma}$	Covariance matrix
$\boldsymbol{\Sigma}$	Covariance matrix
$\hat{(\cdot)}$	Estimate of the mean of $(\cdot)$
$\mathbb{E}(\cdot)$	Mathematical expectation of $(\cdot)$
$\mathbf{A_j}$	Matrix describing the state contribution of the j^{th} contributing analysis, $\mathbf{A_j} \in \mathbb{R}^{l_j \times m}$
$\mathbf{B_j}$	Matrix describing the deterministic input contribution of the j^{th} contributing analysis, $\mathbf{B_j} \in \mathbb{R}^{l_j \times d}$
$\mathbf{C_j}$	Matrix describing the probabilistic input contribution of the j^{th} contributing analysis, $\mathbf{C_j} \in \mathbb{R}^{l_j \times p}$
$\mathbf{d_j}$	Bias associated with the j^{th} contributing analysis, $\mathbf{d_j} \in \mathbb{R}^{l_j}$
$\mathbf{I_{n \times n}}$	The $n \times n$ identity matrix
$\mathbf{Q}$	Covariance of the random noise associated with the model
$\mathbf{R}$	Covariance of the random noise associated with the observation of the system
$\mathbf{u_d}$	Deterministic system-level inputs into the design, $\mathbf{u_d} \in \mathbb{R}^{d}$
$\mathbf{u_p}$	Probabilistic system-level inputs into the design, $\mathbf{u_p} \in \mathbb{R}^{p}$
$\mathbf{v}$	Random noise associated with the observation of the system, $\mathbf{v} \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$
$\mathbf{w}$	Random noise associated with the model, $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$
$\mathbf{y_j}$	Contributing analysis output, $\mathbf{y_j} \in \mathbb{R}^{l_j}$

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$\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$	Normal random variable with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$
$\mathcal{N}(\mu, \sigma^2)$	Normal random variable with mean μ and variance σ^2
$\mathcal{U}(x_{\min}, x_{\max})$	Uniform random variable which varies between $x_{\min}$ and $x_{\max}$
ρ_{X_i,X_j}	Product-moment coefficient between X_i and X_j, $\rho_{X_i,X_j} \in [-1,1]$
$L(\cdot)$	Lagrangian

I. Introduction

By viewing the multidisciplinary design problem as a dynamical system, as is done in Ref. 1–4, a new domain of tools is available to the multidisciplinary design analysis and optimization (MDA/O) community. The MDO problem can be thought of as a multidimensional root-finding problem, which is shown to be a dynamical system.[1] One of the advantageous tools that lends itself for application in MDA/O problems is the use of estimation theory. Through application of estimation theory, a rapid method to obtain characteristics regarding the mean and covariance matrix of the response results. This is achieved through the application of a Kalman filter in a method analogous to that in linear covariance analysis.[5–7] This paper provides the necessary fundamentals to apply this dynamical systems technique to the MDA/O problem and then shows its applicability through two example problems. In addition, a note regarding a bounding technique on the variance which utilizes the covariance matrix is provided.

II. Multidisciplinary Design as a Dynamical System

II.A. Identification of Feasible Designs

Identifying feasible designs in multidisciplinary systems can be thought of as the process of finding the root of a function. Consider a multidisciplinary problem where the analysis variables are described by a multivariable function $\mathbf{f}(\mathbf{u}, \mathbf{p})$ where $\mathbf{u}$ are the design variables and $\mathbf{p}$ are the parameters of the problem. Assume that the requirements of the design are given by only equality constraints that are a function of the performance of the system. The performance of the design is described by a multi-variable mapping $\mathbf{g}(\mathbf{f}(\mathbf{u}, \mathbf{p}))$ and the requirements are given by $\mathbf{z}$. In order to meet the requirements it is necessary to adjust the design variables $\mathbf{u}$ so that

$$\mathbf{z} = \mathbf{g}(\mathbf{f}(\mathbf{u}, \mathbf{p})) \tag{1}$$

The solution $\mathbf{u}^*$ of Eq. (1) is the root of the system. Since identifying feasible designs within the multidisciplinary design problem requires finding the value of $\mathbf{u}$ that satisfies Eq. (1), this process can be thought of as a root-finding problem when an iterative solution method is chosen.

Many numerical methods for finding the root of a function, $\mathbf{g}(\mathbf{u})$, are dynamical systems since they rely on iterative schemes to identify the root.[8] For instance, the bisection method, secant method, function iteration method, and Newton's method are all iterative techniques that satisfy the requirements of a dynamical system.

II.B. Design Optimization

In order for a converged design to be an optimum with respect to some objective function, its performance needs to be evaluated with respect to other potential designs. The general optimization problem is formulated as

$$\left.\begin{array}{lll} \textbf{Minimize:} & \mathcal{J}(\mathbf{u}, \mathbf{p}) & \\ \textbf{Subject to:} & \mathbf{g}_i(\mathbf{u}, \mathbf{p}) \leq \mathbf{0}, & i = 1, \ldots, n_g \\ & \mathbf{h}_j(\mathbf{u}, \mathbf{p}) = \mathbf{0}, & j = 1, \ldots, n_h \\ \textbf{By varying:} & \mathbf{u} & \end{array}\right\} \tag{2}$$

which requires a stationary point of the Lagrangian given as

$$L(\mathbf{u}, \mathbf{p}, \boldsymbol{\lambda}) = \mathcal{J}(\mathbf{u}, \mathbf{p}) + \sum_{i=1}^{n_g} \lambda_i \mathbf{g}_i(\mathbf{u}, \mathbf{p}) + \sum_{j=1}^{n_h} \lambda_{n_g+j} \mathbf{h}_j(\mathbf{u}, \mathbf{p}) \tag{3}$$

to be found. The stationary point of the Lagrangian (Eq. (3)) is the value of $\mathbf{u}$ such that $\nabla_{\mathbf{u}} L(\mathbf{u}, \mathbf{p}, \boldsymbol{\lambda}) = \mathbf{0}$.

II.C. Identifying an Optimal Multidisciplinary Design

Multidisciplinary design optimization can be broken down into two steps: (1) identifying feasible designs and (2) identifying the optimal design from the set of feasible candidates. As discussed, both of these steps are root-finding problems. With the choice of an appropriate iterative numerical root-finding scheme, each of these individual steps can be posed as dynamical systems. When combined together, a nested root-finding problem results.

III. Propagating Uncertainty Using Estimation Theory

Feedback within a multidisciplinary design problem leads to significantly longer analysis times. Several methods have been developed to eliminate feedback within the design. The traditional approach to eliminate the feedback within the design-analysis cycle is to enforce a constraint in the *converged* design that the estimated value of the feedback variable is within a given tolerance of the value resulting from the subsequent CA. This is an effective technique for deterministic analysis and design; however, increasing the number of constraints can be computationally time consuming for robustness assessment and robust design. A novel technique which applies concepts from estimation theory to this challenge is the use of the Kalman filter. This approach is particularly applicable to the robustness analysis problem as the final quantities being sought are the mean and the variance of an objective function. This approach has not been implemented previously because the Kalman filter is typically implemented with respect to a dynamical system and the multidisciplinary analysis and design problem is traditionally concerned with algebraic quantities. This use of the Kalman filter in this fashion is analogous to linear covariance methods described in Refs. 5–7.

IV. The Discrete Kalman Filter

The Kalman filter can be thought of as a two step process, one which predicts the state (*e.g.*, the output of the CAs) and then an update step which corrects these estimates based on the dynamics of the system. The prediction step is given by the following equations[5,9–14]

$$\hat{\mathbf{y}}_{k|k-1} = \mathbf{F}_k \hat{\mathbf{y}}_{k-1|k-1} + \mathbf{B}_k \mathbf{u}_k \tag{4}$$
$$\mathbf{\Sigma}_{k|k-1} = \mathbf{F}_k \mathbf{\Sigma}_{k-1|k-1} \mathbf{F}_k^T + \mathbf{Q}_k \tag{5}$$

where the notation $j|k$ represents the estimate at j given observations up to and including k. Furthermore, the value of $\hat{\mathbf{y}}_{0|0}$ is the initial mean state and $\mathbf{\Sigma}_{0|0}$ is the initial covariance matrix of the state values. The correction step is governed by the following equations[5,9–14]

$$\tilde{\mathbf{x}}_k = \mathbf{z}_k - \mathbf{H}_k \hat{\mathbf{y}}_{k|k-1} \tag{6}$$
$$\mathbf{S}_k = \mathbf{H}_k \mathbf{\Sigma}_{k|k-1} \mathbf{H}_k^T + \mathbf{R}_k \tag{7}$$
$$\mathbf{K}_k = \mathbf{\Sigma}_{k|k-1} \mathbf{H}_k^T \mathbf{S}_k^{-1} \tag{8}$$
$$\hat{\mathbf{y}}_{k|k} = \hat{\mathbf{y}}_{k|k-1} + \mathbf{K}_k \tilde{\mathbf{x}}_k \tag{9}$$
$$\mathbf{\Sigma}_{k|k} = \left(\mathbf{I} - \mathbf{K}_k \mathbf{H}_k\right) \mathbf{\Sigma}_{k|k-1} \tag{10}$$

where the final (*a posteriori*) estimate of the state is given by $\hat{\mathbf{y}}_{k|k}$ with covariance matrix given by $\mathbf{\Sigma}_{k|k}$.

V. Formulating the Multidisciplinary Design Problem in a Form Compatible with the Kalman Filter

The root-finding problem has been shown to be a dynamical system which can be defined by the relation

$$\mathbf{y}_k = \mathbf{f}(\mathbf{y}_{k-1}), \quad \forall k \in \mathbb{Z}_+ \setminus \{0\} \tag{11}$$

where $\mathbf{f}(\mathbf{y}_{k-1})$ is the output value of the CAs on the $k^{\text{th}} - 1$ iteration. For random variables in a linear system, this can be written in the form

$$\mathbf{y}_k = \mathbf{F}_k \mathbf{y}_{k-1} + \mathbf{w}_{k-1}, \quad \forall k \in \mathbb{Z}_+ \setminus \{0\} \tag{12}$$

where $\mathbf{w}_{k-1}$ is the noise associated with the model. For a linear multidisciplinary design, Eq. (12) can also be written as

$$\mathbf{y}_k = \mathbf{F}_k \mathbf{y}_{k-1} + \mathbf{B}_k \mathbf{u}_k + \mathbf{w}_{k-1} \tag{13}$$

which allows for inputs into the CA that are not outputs of other CAs, $\mathbf{u}_k$. When coupled with an equation of the form

$$\mathbf{z}_k = \mathbf{H}_k \mathbf{y}_k + \mathbf{v}_k \tag{14}$$

and when it is assumed that $\mathbf{w}_{k-1} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_{k-1})$ and $\mathbf{v}_{k-1} \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_{k-1})$, Eqs. (13) and (14) define the dynamical system needed for a Kalman filter.[10–13] The noise parameter, $\mathbf{w}_{k-1}$, gives the opportunity to account for random variables within the linearization of the input-output relationship, that is random variables associated with the matrix $\mathbf{F}$. In this work, the Kalman filter is used as a data fusion technique to give an optimal unbiased statistical estimate of the output of the CAs as the design is converging.

The power in implementing the Kalman filter in multidisciplinary design analysis lies in the ability to obtain a continuous estimate in iterate of both the mean and covariance of each CA in the multidisciplinary design by propagating a system of seven equations until the design converges.

VI. Using the Covariance Matrix to Guide Design Decomposition

As one of the outputs of the Kalman filter is the estimated covariance at iteration k this information could be used to ascertain the correlation coefficient between variables. The covariance estimate, $\mathbf{\Sigma}$, has the form

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma^2_{X_1} & \rho_{X_1,X_2}\sigma_{X_1}\sigma_{X_2} & \cdots & \rho_{X_1,X_2}\sigma_{X_1}\sigma_{X_n} \\ \rho_{X_1,X_2}\sigma_{X_2}\sigma_{X_1} & \sigma^2_{X_2} & \cdots & \rho_{X_2,X_n}\sigma_{X_2}\sigma_{X_n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{X_1,X_n}\sigma_{X_n}\sigma_{X_1} & \rho_{X_2,X_n}\sigma_{X_n}\sigma_{X_2} & \cdots & \sigma^2_{X_n} \end{pmatrix} \tag{15}$$

where $\sigma^2_{X_i}$ is the variance of variable X_i and ρ_{X_i,X_j} is the product-moment coefficient (*i.e.*, the correlation coefficient) given by

$$\rho_{X_i,X_j} = \frac{\mathbb{E}\left[(X_i - \mu_{X_i})(X_j - \mu_{X_j})\right]}{\sigma_{X_i}\sigma_{X_j}} \tag{16}$$

Alternatively, the covariance matrix can be represented as

$$\mathbf{\Sigma} = \{\mathbf{\Sigma}_{ij}\} \tag{17}$$

then the representative correlation (or product-moment) coefficients are given by

$$\rho_{ij} = \frac{\mathbf{\Sigma}_{ij}}{\sqrt{\mathbf{\Sigma}_{ii}}\sqrt{\mathbf{\Sigma}_{jj}}}, \quad i \neq j \tag{18}$$

where $\rho_{ij} \in [-1, 1]$. What is important about the correlation coefficient is that it gives a relative measure of how variable j depends on variable i. In particular as $|\rho_{ij}| \to 1$ the importance of variable i on the response of variable j increases. This gives a meaningful way to ascertain the importance of each CA and variables on other CAs and variables. For design decomposition, it may be acceptable to neglect the feedback variables with small correlation coefficient magnitudes.

VII. Use of Estimation Theory in Multidisciplinary Design

In Refs. 1–3, a methodology that rapidly obtains the mean and a bound on the variance of a multidisciplinary design was developed. This new methodology treats the multidisciplinary design problem as a dynamical system. Viewing the multidisciplinary design problem as a dynamical system enables stability, control, and estimation techniques from dynamical system theory to be applied in order to rapidly obtain a robust optimal design.[1]

VII.A. Procedure

Ref. 1 provides a rigorous description and derivation of the steps of the rapid robust design methodology which are summarized below.

1. **Decompose the design**: A general multidisciplinary design can be decomposed into multiple contributing analyses (CAs). Each of these CAs represents an analysis that contributes to the entire design. In the theoretical development underlying Ref. 1, it is assumed that each of the CAs are linear and algebraic, where the output of each of the CAs is of the form

$$\mathbf{y_j} = \mathbf{A_j}\mathbf{y} + \mathbf{B_j}\mathbf{u_d} + \mathbf{C_j}\mathbf{u_p} + \mathbf{d_j} \tag{19}$$

   where $\mathbf{y}$ is the concatenated output from all of the CAs, $\mathbf{u_d}$ are the deterministic system-level inputs into the design, $\mathbf{u_p}$ are the probabilistic system-level inputs into the design, and $\mathbf{d_j}$ is the bias associated with the model. For general designs where the CAs may not be linear, the required functional form can be achieved through linearization.

2. **Identify the random variables in the design and their distributions**: To propagate the uncertainties associated with the parameters of the design through the design to estimate the robustness, the probabilistic variables must be identified. The random variables associated with the uncertainty within the design are handled in two different ways depending on where the random variable is functionally located. If the uncertainties are associated with the parameters of the design they are propagated through the covariance estimation of the Kalman filter. On the other hand, if the uncertainties are associated with the CAs, they are propagated through the bias term of the filter equations.

3. **Form the iterative equations**: In order to implement this methodology, a causal, discrete dynamical system must be formed. Assuming fixed-point iteration is used to converge the design, the dynamical system is given by

$$\mathbf{y}_k = \mathbf{\Lambda}\mathbf{y}_{k-1} + \boldsymbol{\beta}\mathbf{u_d} + \boldsymbol{\gamma}\mathbf{u_p} + \boldsymbol{\delta} \tag{20}$$

   where $\mathbf{\Lambda} = \begin{pmatrix}\mathbf{A_1}^T & \cdots & \mathbf{A_n}^T\end{pmatrix}^T$, $\boldsymbol{\beta} = \begin{pmatrix}\mathbf{B_1}^T & \cdots & \mathbf{B_n}^T\end{pmatrix}^T$, $\boldsymbol{\gamma} = \begin{pmatrix}\mathbf{C_1}^T & \cdots & \mathbf{C_n}^T\end{pmatrix}^T$, and $\boldsymbol{\delta} = \begin{pmatrix}\mathbf{d_1}^T & \cdots & \mathbf{d_n}^T\end{pmatrix}^T$.

4. **Estimate the mean output and the covariance of the design**: The mean output of the multidisciplinary system and the associated covariance matrix are found by propagating the Kalman filter equations until convergence. In order to accomplish this, the iterative system formed in Eq. (20) needs to be transformed to the form needed by the Kalman filter by making the following substitutions

$$\mathbf{F}_{k-1} = \mathbf{\Lambda}, \quad \forall k \in \{1, 2, ...\} \tag{21}$$

$$\mathbf{B}_{k-1} = \begin{pmatrix}\boldsymbol{\beta} & \boldsymbol{\gamma} & \mathbf{I_{m\times m}}\end{pmatrix}, \quad \forall k \in \{1, 2, ...\} \tag{22}$$

$$\mathbf{u}_{k-1} = \begin{pmatrix}\mathbf{u_d}^T & \mathbf{u_p}^T & \boldsymbol{\delta}^T\end{pmatrix}^T, \quad \forall k \in \{1, 2, ...\} \tag{23}$$

   The iterates are then found by by propagating the filter equations until the design convergence criterion is met.

5. **Identify the mean and variance bound of the objective function**: The mean and variance of the design objective are found from the last iterate of the Kalman filter propagation. Where the mean value is taken directly from the filter value and a bound on the variance is found by applying the matrix 2-norm to the covariance matrix.

VIII. Robustness Estimate in a Linear, Two Contributing Analysis Design

To show the accuracy of the mean and variance estimate provided by the Kalman filter in multidisciplinary design, consider the coupled, linear two CA system shown in Fig. 1.
For this analysis, assume that there are two components to the probabilistic parameter vector and the two output vectors, that is $\mathbf{u_p} \in \mathbb{R}^2$, $\mathbf{y_1} \in \mathbb{R}^2$, and $\mathbf{y_2} \in \mathbb{R}^2$, which, in turn, implies $\mathbf{A'_1} \in \mathbb{R}^{2\times 2}$, $\mathbf{B'_1} \in \mathbb{R}^2$,

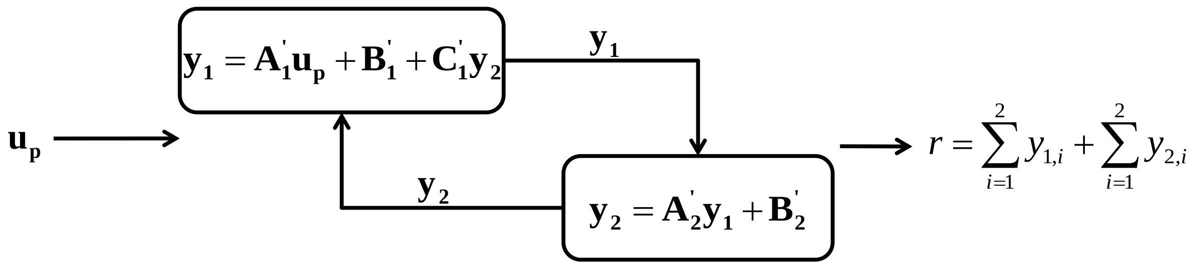

Figure 1. Two-contributing analysis multidisciplinary design.

$\mathbf{C_1'} \in \mathbb{R}^{2\times 2}$, $\mathbf{A_2'} \in \mathbb{R}^{2\times 2}$, and $\mathbf{B_2'} \in \mathbb{R}^2$. Also, let the distribution of the probabilistic parameter input be given by a multivariate normal, $\mathbf{u_p} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{u_p}}, \boldsymbol{\Sigma}_{\mathbf{u_p}})$.

The effectiveness of using estimation theory in multidisciplinary design to obtain robustness estimates will be demonstrated by letting the mean of the probabilistic input, $\boldsymbol{\mu}_{\mathbf{u_p}}$, and the components of the covariance matrix $\sigma^2_{y_1}$, $\sigma^2_{y_2}$, and $\rho_{y_1 y_2}$ vary between given ranges. The maximum error between the response obtained from the robustness assessment methodology and an analytical propagation is then reported for a multitude of points within the design space.

VIII.A. Analytical Solution

As this is a multidisciplinary analysis consisting of two linear CAs, there is a single simultaneous solution for $\mathbf{y_1}$ and $\mathbf{y_2}$ which is found to be

$$\left.\begin{array}{rcl} \mathbf{y_1} & = & \left(\mathbf{I_{2\times 2}} - \mathbf{C_1'A_2'}\right)^{-1}\left(\mathbf{A_1'u_p} + \mathbf{B_1'} + \mathbf{C_1'B_2'}\right) \\ \mathbf{y_2} & = & \mathbf{A_2'}\left(\mathbf{I_{2\times 2}} - \mathbf{C_1'A_2'}\right)^{-1}\left(\mathbf{A_1'u_p} + \mathbf{B_1'} + \mathbf{C_1'B_2'}\right) + \mathbf{B_2'} \end{array}\right\}$$

which implies that whenever $\mathbf{I_{2\times 2}} - \mathbf{C_1'A_2'}$ is non-singular, a unique solution exists for $\mathbf{y_1}$ and $\mathbf{y_2}$. Since the only uncertainty in this analysis is given by the probabilistic input vector, $\mathbf{u_p}$, which is defined as a multivariate normal, the distribution of the output for each CA can be found exactly. These are given by

$$\left.\begin{array}{rcl} \mathbf{y_1} & \sim & \mathcal{N}\left(\boldsymbol{\mu}_{\mathbf{y_1}}, \boldsymbol{\Sigma}_{\mathbf{y_1}}\right) \\ \mathbf{y_2} & \sim & \mathcal{N}\left(\boldsymbol{\mu}_{\mathbf{y_2}}, \boldsymbol{\Sigma}_{\mathbf{y_2}}\right) \end{array}\right\}$$

where

$$\begin{array}{rcl} \boldsymbol{\mu}_{\mathbf{y_1}} & = & \left(\mathbf{I_{2\times 2}} - \mathbf{C_1'A_2'}\right)^{-1}\mathbf{A_1'}\boldsymbol{\mu}_{\mathbf{u_p}} + \left(\mathbf{I_{2\times 2}} - \mathbf{C_1'A_2'}\right)^{-1}\left(\mathbf{B_1'} + \mathbf{C_1'B_2'}\right) \\ \boldsymbol{\Sigma}_{\mathbf{y_1}} & = & \left(\mathbf{I_{2\times 2}} - \mathbf{C_1'A_2'}\right)^{-1}\mathbf{A_1'}\boldsymbol{\Sigma}_{\mathbf{u_p}}\mathbf{A_1'}^T\left(\mathbf{I_{2\times 2}} - \mathbf{C_1'A_2'}\right)^{-T} \end{array}$$

and

$$\begin{array}{rcl} \boldsymbol{\mu}_{\mathbf{y_2}} & = & \mathbf{A_2'}\left(\mathbf{I_{2\times 2}} - \mathbf{C_1'A_2'}\right)^{-1}\mathbf{A_1'}\boldsymbol{\mu}_{\mathbf{u_p}} + \mathbf{A_2'}\left(\mathbf{I_{2\times 2}} - \mathbf{C_1'A_2'}\right)^{-1}\left(\mathbf{B_1'} + \mathbf{C_1'B_2'}\right) + \mathbf{B_2'} \\ \boldsymbol{\Sigma}_{\mathbf{y_2}} & = & \mathbf{A_2'}\left(\mathbf{I_{2\times 2}} - \mathbf{C_1'A_2'}\right)^{-1}\mathbf{A_1'}\boldsymbol{\Sigma}_{\mathbf{u_p}}\mathbf{A_1'}^T\left(\mathbf{I_{2\times 2}} - \mathbf{C_1'A_2'}\right)^{-T}\mathbf{A_2'}^T \end{array}$$

Since both of the output distributions from the CAs are also multivariate normal, the components of the response

$$r = \sum_{i=1}^{2} y_{1,i} + \sum_{i=1}^{2} y_{2,i}$$

can be found exactly by summing the components of mean components of $\boldsymbol{\mu}_{\mathbf{y_1}}$ and $\boldsymbol{\mu}_{\mathbf{y_2}}$ to find the mean of the response and adding the appropriate variances from the covariance matrices $\boldsymbol{\Sigma}_{\mathbf{y_1}}$ and $\boldsymbol{\Sigma}_{\mathbf{y_2}}$. That is

$$r \sim \mathcal{N}\left(\sum_{i=1}^{2}\boldsymbol{\mu}_{\mathbf{y_1},i} + \sum_{i=1}^{2}\boldsymbol{\mu}_{\mathbf{y_2},i}, \sum_{i=1}^{2}\lambda(\boldsymbol{\Sigma}_{\mathbf{y_1}})|_i + \sum_{i=1}^{2}\lambda(\boldsymbol{\Sigma}_{\mathbf{y_2}})|_i\right)$$

where $\boldsymbol{\mu}_{\mathbf{y_1},i}$ is the i^{th} component of $\mathbf{y_1}$, $\boldsymbol{\mu}_{\mathbf{y_2},i}$ is the i^{th} component of $\mathbf{y_2}$, and $\lambda(\cdot)|_i$ is the i^{th} eigenvalue of the matrix argument.

VIII.B. Obtaining a Robustness Estimate

Step 1: Decompose the Design
The problem as given has already been decomposed into the representative contributing analyses; however, it is still necessary to identify each of the terms in Eq. (19). For the first CA, $\mathbf{y_1}$, the functional form is as follows

$$\mathbf{y_1} = \begin{pmatrix} \mathbf{0} & \mathbf{C'_1} \end{pmatrix} \mathbf{y} + \begin{pmatrix} \mathbf{0} \end{pmatrix} \mathbf{u_d} + \begin{pmatrix} \mathbf{A'_1} \end{pmatrix} \mathbf{u_p} + \mathbf{B'_1}$$

Similarly, for the second CA, the functional form is given by

$$\mathbf{y_2} = \begin{pmatrix} \mathbf{A'_2} & \mathbf{0} \end{pmatrix} \mathbf{y} + \begin{pmatrix} \mathbf{0} \end{pmatrix} \mathbf{u_d} + \begin{pmatrix} \mathbf{0} \end{pmatrix} \mathbf{u_p} + \mathbf{B'_2}$$

Hence,

$$\mathbf{A_1} = \begin{pmatrix} \mathbf{0} & \mathbf{C'_1} \end{pmatrix} \quad \mathbf{B_1} = \begin{pmatrix} \mathbf{0} \end{pmatrix}$$

$$\mathbf{C_1} = \begin{pmatrix} \mathbf{A'_1} \end{pmatrix} \quad \mathbf{d_1} = \mathbf{B'_1}$$

$$\mathbf{A_2} = \begin{pmatrix} \mathbf{A'_2} & \mathbf{0} \end{pmatrix} \quad \mathbf{B_2} = \begin{pmatrix} \mathbf{0} \end{pmatrix}$$

$$\mathbf{C_2} = \begin{pmatrix} \mathbf{0} \end{pmatrix} \quad \mathbf{d_2} = \mathbf{B'_2}$$

Step 2: Identify the Random Variables and their Distributions
There is only one set of random variables in this example, that of the probabilistic input variable, $\mathbf{u_p}$. This is given in the problem description as a multivariate normal distribution, $\mathbf{u_p} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{u_p}}, \boldsymbol{\Sigma}_{\mathbf{u_p}})$. Later, the two defining parameters of the multivariate normal will be given numerical values.

Step 3: Form the Iterative Equations
In order to use the Kalman filter to simultaneously estimate the robustness and converge the design, the iterative equations described in Eq. (20) for fixed-point iteration need to be formed. Through analogy of variables, the matrices are given by

$$\boldsymbol{\Lambda} = \begin{pmatrix} \mathbf{A_1} \\ \mathbf{A_2} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{C'_1} \\ \mathbf{A'_2} & \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{\beta} = \begin{pmatrix} \mathbf{B_1} \\ \mathbf{B_2} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{\gamma} = \begin{pmatrix} \mathbf{C_1} \\ \mathbf{C_2} \end{pmatrix} = \begin{pmatrix} \mathbf{A'_1} \\ \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{\delta} = \begin{pmatrix} \mathbf{d_1} \\ \mathbf{d_2} \end{pmatrix} = \begin{pmatrix} \mathbf{B'_1} \\ \mathbf{B'_2} \end{pmatrix}$$

Step 4: Estimate the Mean Output and the Covariance
The equations formed in the prior step can then be propagated through the Kalman filter defined by Eqs. (4)-(10) with

$$\mathbf{F}_{k-1} = \boldsymbol{\Lambda}, \quad \forall k \in \{1, 2, ...\}$$

$$\mathbf{B}_{k-1} = \begin{pmatrix} \boldsymbol{\beta} & \boldsymbol{\gamma} & \mathbf{I_{4\times4}} \end{pmatrix}, \quad \forall k \in \{1, 2, ...\}$$

$$\mathbf{u}_{k-1} = \begin{pmatrix} \mathbf{u_d} \\ \mathbf{u_p} \\ \delta \end{pmatrix}, \quad \forall k \in \{1, 2, ...\}$$

where in this example $\mathbf{u_d} = \mathbf{0}$ and $\mathbf{u_p} = \mathbb{E}(\mathbf{u_p}) = \boldsymbol{\mu}_{\mathbf{u_p}}$. In this example, the matrix $\mathbf{Q}$ is the null matrix since the only uncertain parameters of the problem are associated with the input parameters, not the model. The unscented transform is used on an uncoupled system with the distribution described in Step 2 in order to identify $\mathbf{y_0}$ and $\mathbf{\Sigma_0}$, the initial output mean and covariance for each design. A design is considered converged when the absolute difference between iteration estimates is less than 1×10^{-4} or the relative difference is less than 1×10^{-6}.

Step 5: Identify the Mean and Variance Bound of the Objective Function

Upon convergence, the value of $\mathbf{y}$, the state variable in the problem, is the mean response for each of the components of the output CAs. In this example,

$$\bar{r} = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \hat{\mathbf{y}}_{n|n}$$

The estimate for the variance (*i.e.*, the variance bound) in this case is

$$\sigma_r^2 \leq \sum_{i=1}^{4} \| \mathbf{\Sigma}_{\mathbf{y^*}_{n|n}} \|_2 = 4 \| \mathbf{\Sigma}_{\mathbf{y^*}_{n|n}} \|_2$$

.

VIII.C. Analysis Results

In order to assess a large variety of problems, a parametric sweep of the design variables was performed to identify the maximum errors in the design space. To perform this parameter sweep, the problem's parameters were varied independently as shown in Table 1 where the distribution of each variable was assumed to be uniform and a 100,000 case Monte Carlo analysis was conducted.

Table 1. Parameter ranges to assess the validity of the rapid robust design methodology.

Parameter	Distribution
σ_1^2	$\mathcal{U}(0, 100)$
σ_2^2	$\mathcal{U}(0, 100)$
$\rho_{y_1 y_2}$	$\mathcal{U}(-1, 1)$
$\mathbf{A}_1'$	$\begin{pmatrix} \mathcal{U}(-1,1) & \mathcal{U}(-1,1) \\ \mathcal{U}(-1,1) & \mathcal{U}(-1,1) \end{pmatrix}$
$\mathbf{B}_1'$	$\begin{pmatrix} \mathcal{U}(-1,1) \\ \mathcal{U}(-1,1) \end{pmatrix}$
$\mathbf{C}_1'$	$\begin{pmatrix} \mathcal{U}(-1,1) & \mathcal{U}(-1,1) \\ \mathcal{U}(-1,1) & \mathcal{U}(-1,1) \end{pmatrix}$
$\mathbf{A}_2'$	$\begin{pmatrix} \mathcal{U}(-1,1) & \mathcal{U}(-1,1) \\ \mathcal{U}(-1,1) & \mathcal{U}(-1,1) \end{pmatrix}$
$\mathbf{B}_2'$	$\begin{pmatrix} \mathcal{U}(-1,1) \\ \mathcal{U}(-1,1) \end{pmatrix}$

In order to guarantee convergence of the design, constraints were imposed on the parameters to ensure that all of the eigenvalues of the matrix $\mathbf{\Lambda}$ had modulus less than unity. To ensure realizable covariance matrices, that is a matrix that is symmetric and positive definite, the components of the covariance (*e.g.*, variance and correlation coefficient) were determined independently and then combined to form the covariance matrix.

In addition to the parameters shown in Table 1, the effect of the mean of the probabilistic parameters was conducted by analyzing three different cases–one where the mean was $\boldsymbol{\mu}_{\mathbf{u_p}} = (0\ 0)^T$, one where the mean was $\boldsymbol{\mu}_{\mathbf{u_p}} = (100\ 0)^T$, and one where the mean was $\boldsymbol{\mu}_{\mathbf{u_p}} = (100\ 100)^T$. The results were then compared with results propagated analytically resulting in Figs. 2-4.

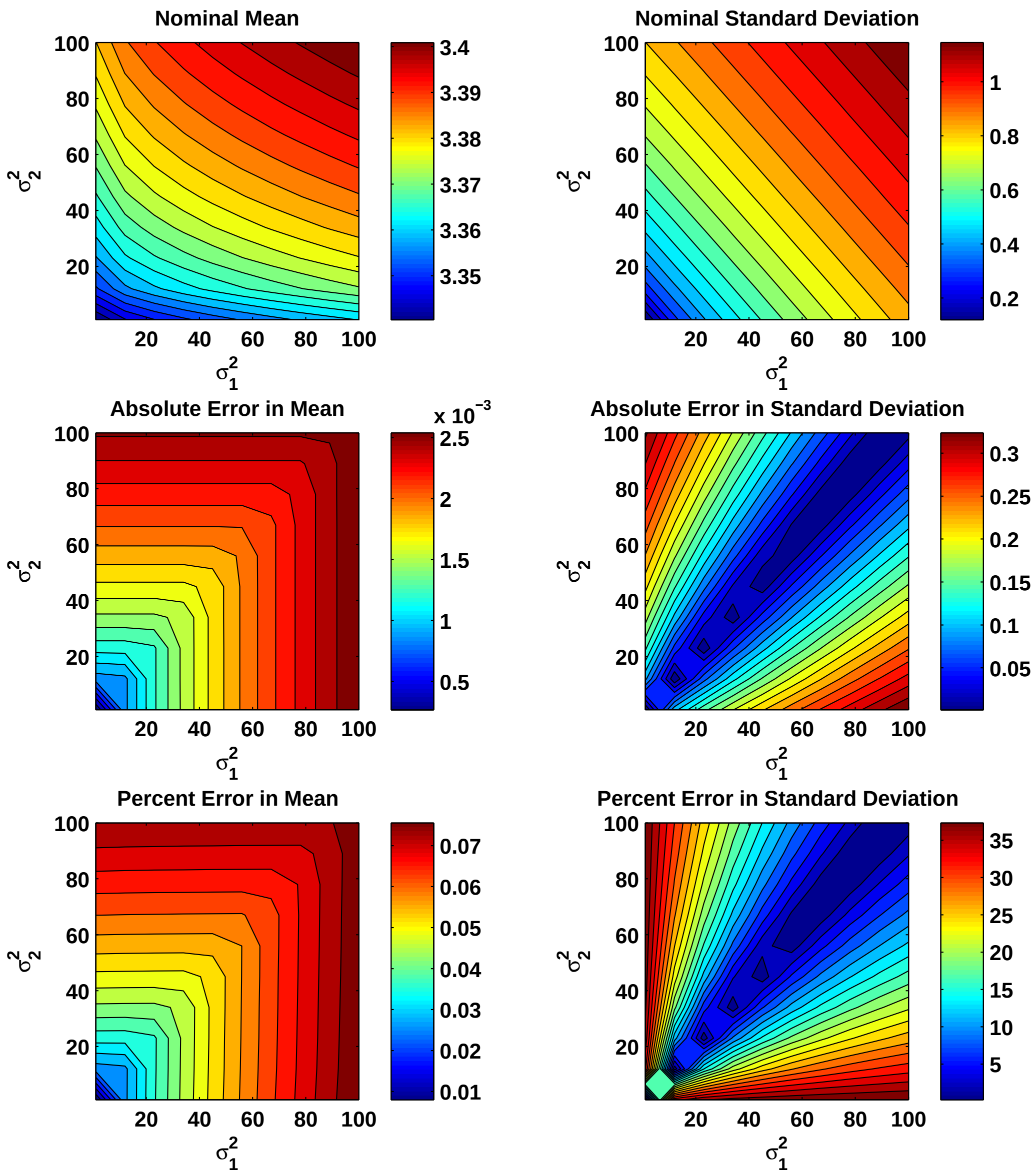

Figure 2. Maximum error for a two contributing analysis multidisciplinary design with $\mu_{\mathbf{u_p}} = (0\ 0)^T$.

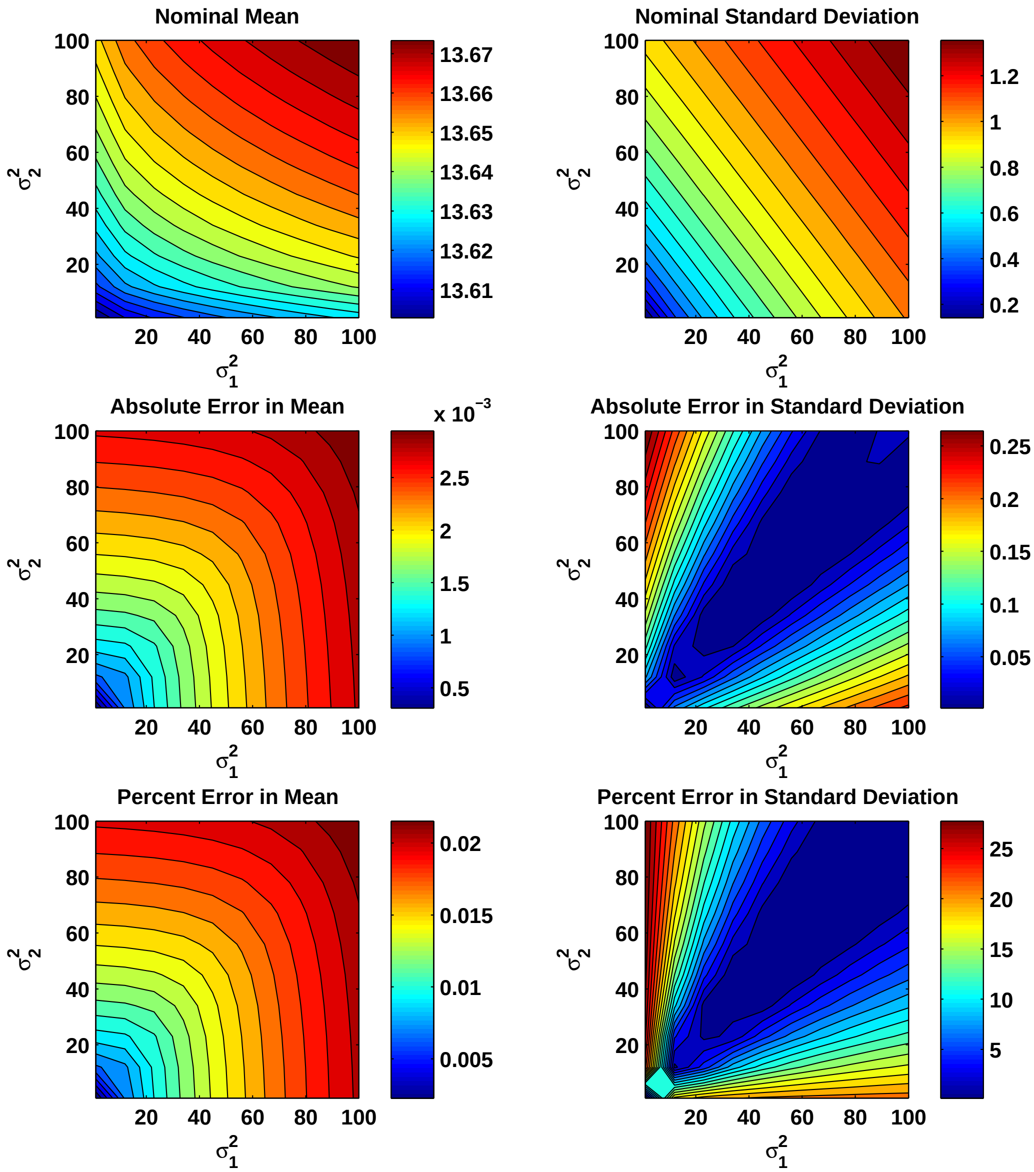

Figure 3. Maximum error for a two contributing analysis multidisciplinary design with $\mu_{\mathbf{u_p}} = (100\ 0)^T$.

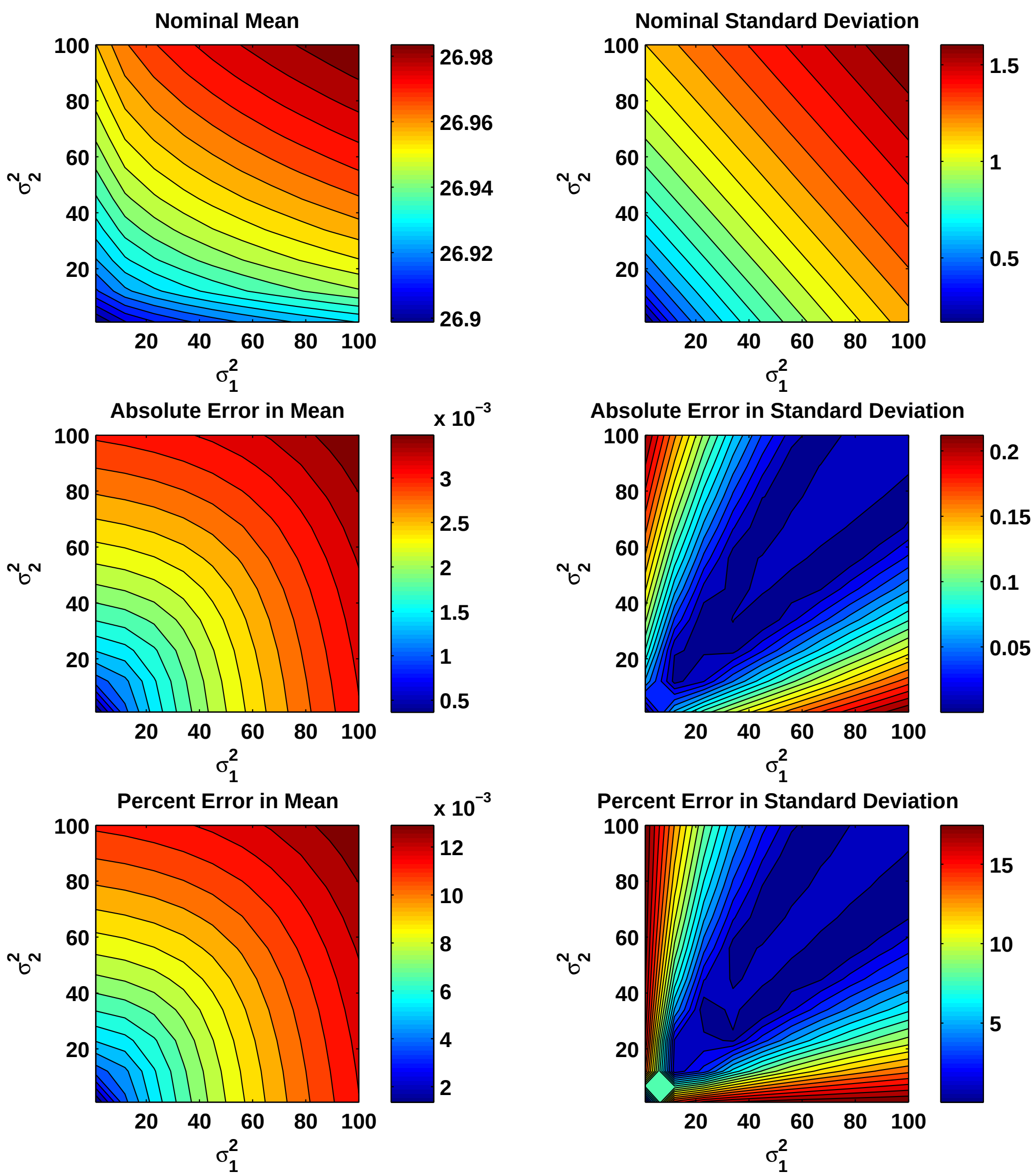

Figure 4. Maximum error for a two contributing analysis multidisciplinary design with $\boldsymbol{\mu}_{\mathbf{up}} = (100\ 100)^T$.

It is observed from these results that the mean error is less than 0.08% for all of the cases examined. This is a result of the system being linear and the Kalman filter propagating results exactly for a linear system. Therefore the error in the mean is solely a result of the convergence criterion being utilized. For each case, there is seen to be a rise in the standard deviation error near the origin. This is because the nominal mean goes to zero causing a rise in the in the percent error near this point.

Using estimation techniques is observed to provide a consistent conservative bound on the variance as seen in Figs. 2-4 since all of the percent error values are positive. It is also interesting to note that the error in mean and standard deviation, regardless of the mean of the input, appears to be close to the same order of magnitude. As the mean input value increases, the magnitude of the mean response and standard deviation of that response increases, which causes a decrease in the percent error. Furthermore, it is observed that the maximum error approaches a limit of less than 40%. This limit is a function of the two-norm being used. This limit is described and related to the dimensionality of the problem subsequently. In analyzing the data, the largest errors are caused for weakly coupled systems, that is systems where $\mathbf{C_1'}$ is small. This can be explained since $\mathbf{C_1'}$ being small leads to a larger domain of values that lead to a converged design. Additionally, since the interplay between $\mathbf{y_1}$ and $\mathbf{y_2}$ is reduced, the iterations to achieve convergence is reduced in these cases.

IX. Analysis of a Two Bar Truss

Consider the planar truss which consists of two elements with a vertical load at the mutual joint, as shown in Fig. 5 (adapted from Ref. 15).

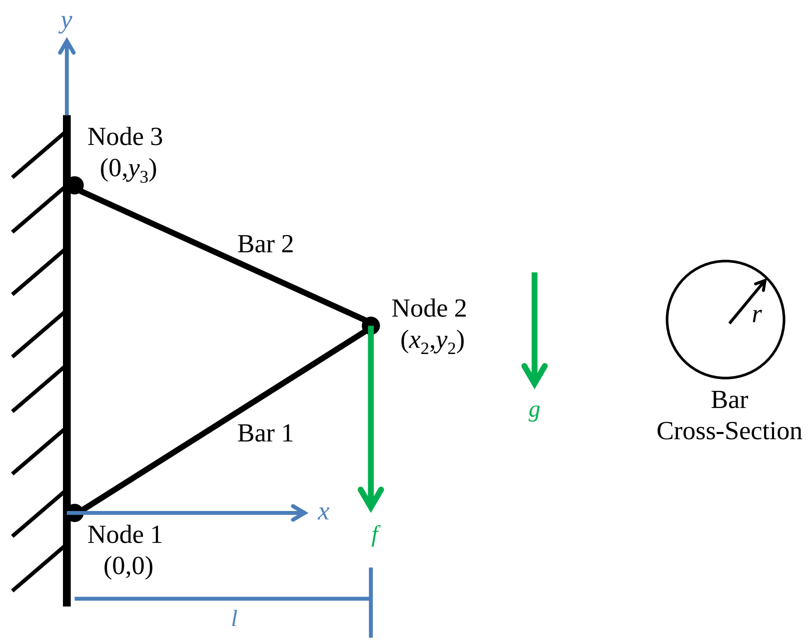

Figure 5. Two bar truss with a load at the mutual joint.

For this problem, it is desired to find the analyze various vertical position of nodes 2 and 3, h_2 and h_3, while ensuring that the structure will not fail due to Euler buckling or yielding with some factor of safety given fixed values for the material properties, E, σ_y, and ρ, the load, f, and the bar geometry, r_1 and r_2. The horizontal position of node 2 is constrained to be l.

Table 2. Parameters for the two-bar truss problem.

Parameter	Description	Nominal Value	Distribution
E	Young's Modulus	200×10^6 kN/m^2	–
σ_y	Yield Strength	250×10^3 kN/m^2	$\mathcal{N}(250 \times 10^3, 625 \times 10^6)$
ρ	Density	7850 kg/m^3	$\mathcal{N}(7850, 100)$
l	Length	5 m	–
r_1	Radius of Bar 1	30 mm	–
r_2	Radius of Bar 2	5 mm	–
f	Applied Force	3.5 kN	$\mathcal{N}(3.5, 0.49)$
g	Gravitational Acceleration	9.81 m/s^2	–

IX.A. Formulation of the Governing Equations

Step 1: Decompose the Design

Two analyses must occur in order to design the two bar truss: a structural analysis and a sizing of the bars constituting the truss. Although not explicit in the problem statement, the mass of the bars also provide a load through their weight. Hence, this is a coupled analysis problem since the structural analysis depends on the sizing of each of the bars. The coupled DSM is shown in Fig. 6.

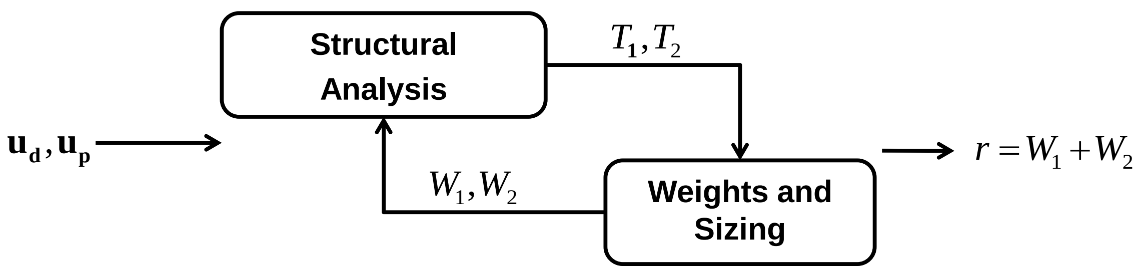

Figure 6. Two bar truss design structure matrix.

The inputs into the design problem are the deterministic and probabilistic parameters of the problem whose values are shown in Table 2. In particular,

$$\mathbf{u_d} = \begin{pmatrix} E & l & r_1 & r_2 & g & y_2 & y_3 \end{pmatrix}^T$$

and

$$\mathbf{u_p} = \begin{pmatrix} \sigma_y & \rho & f \end{pmatrix}^T$$

The structural analysis CA feeds the forces seen in each of the members of the truss to the weights and sizing module. These can be found through the static equilibrium equations and are found by solving the linear equations

$$\begin{pmatrix} \frac{l}{L_1} & 0 & 1 & 0 & 0 & 0 \\ -\frac{y_2}{L_1} & 0 & 0 & 1 & 0 & 0 \\ -\frac{l}{L_1} & \frac{l}{L_2} & 0 & 0 & 0 & 0 \\ -\frac{y_2}{L_1} & \frac{y_3 - y_2}{L_2} & 0 & 0 & 0 & 0 \\ 0 & \frac{l}{L_2} & 0 & 0 & 1 & 0 \\ 0 & \frac{y_3 - y_2}{L_2} & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ R_{1x} \\ R_{1y} \\ R_{3x} \\ R_{3y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{l}{2y_3}(W_1 + W_2 + 2f) \\ f\left(1 + \frac{y_2}{y_3}\right) \\ 0 \\ 0 \\ f \\ \frac{l}{2y_3}(W_1 + W_2 + 2f) \\ f\left(1 + \frac{y_2}{y_3}\right) + W_1 + W_2 \end{pmatrix}$$

for the tensions. The weights and sizing CA computes the weights of each of the bars based on the relationship

$$\mathbf{y_2} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} \pi \rho g r_1^2 L_1 \\ \pi \rho g r_2^2 L_2 \end{pmatrix}$$

Both relationships defined by the CAs rely on the lengths of the bars, which are given by

$$\begin{aligned} L_1 &= \sqrt{l^2 + y_2^2} \\ L_2 &= \sqrt{l^2 + (y_3 - y_2)^2} \end{aligned}$$

This is a realistic example in which the CAs are nonlinear. Therefore, in order to apply the developed methodology, a Taylor series expansion about a nominal value (chosen to be the previous iterate's mean value) must be conducted. Functionally, this means that Eq. (IX.A) can be expanded as follows

$$\mathbf{y_1} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \approx \begin{pmatrix} \left.\frac{\partial T_1}{\partial \mathbf{u_d}}\right|_{\hat{\mathbf{u}}_\mathbf{d}} (\mathbf{u_d} - \hat{\mathbf{u}}_\mathbf{d}) + \left.\frac{\partial T_1}{\partial \mathbf{u_p}}\right|_{\boldsymbol{\mu}_{\mathbf{u_p}}} (\mathbf{u_p} - \boldsymbol{\mu}_{\mathbf{u_p}}) + \left.\frac{\partial T_1}{\partial \mathbf{y}}\right|_{\hat{\mathbf{y}}} (\mathbf{y} - \hat{\mathbf{y}}) \\ \left.\frac{\partial T_2}{\partial \mathbf{u_d}}\right|_{\hat{\mathbf{u}}_\mathbf{d}} (\mathbf{u_d} - \hat{\mathbf{u}}_\mathbf{d}) + \left.\frac{\partial T_2}{\partial \mathbf{u_p}}\right|_{\boldsymbol{\mu}_{\mathbf{u_p}}} (\mathbf{u_p} - \boldsymbol{\mu}_{\mathbf{u_p}}) + \left.\frac{\partial T_2}{\partial \mathbf{y}}\right|_{\hat{\mathbf{y}}} (\mathbf{y} - \hat{\mathbf{y}}) \end{pmatrix}$$

Similarly, Eq. (IX.A) can be expanded as

$$\mathbf{y_2} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \approx \begin{pmatrix} \pi \rho g r_1^2 \frac{\hat{y}_2}{\sqrt{l^2 + \hat{y}_2^2}} (y_2 - \hat{y}_2) \\ \pi \rho g r_2^2 \left(\frac{\hat{y}_2 - \hat{y}_3}{\sqrt{l^2 + (\hat{y}_2 - \hat{y}_3)^2}} (y_2 - \hat{y}_2) + \frac{\hat{y}_3 - \hat{y}_2}{\sqrt{l^2 + (\hat{y}_2 - \hat{y}_3)^2}} (y_3 - \hat{y}_3) \right) \end{pmatrix}$$

Therefore, in the form of Eq. (19)

$$\mathbf{A_1} = \left(\left.\frac{\partial T_1}{\partial \mathbf{y}}\right|_{\hat{\mathbf{y}}} \right) \quad \mathbf{B_1} = \left(\left.\frac{\partial T_1}{\partial \mathbf{u_d}}\right|_{\hat{\mathbf{u}}_\mathbf{d}} \right)$$

$$\mathbf{C_1} = \left(\left.\frac{\partial T_1}{\partial \mathbf{u_p}}\right|_{\boldsymbol{\mu}_{\mathbf{u_p}}} \right)$$

$$\mathbf{d_1} = -\left(\left.\frac{\partial T_1}{\partial \mathbf{u_d}}\right|_{\hat{\mathbf{u}}_\mathbf{d}} \hat{\mathbf{u}}_\mathbf{d} + \left.\frac{\partial T_1}{\partial \mathbf{u_p}}\right|_{\boldsymbol{\mu}_{\mathbf{u_p}}} \boldsymbol{\mu}_{\mathbf{u_p}} + \left.\frac{\partial T_1}{\partial \mathbf{y}}\right|_{\hat{\mathbf{y}}} \hat{\mathbf{y}} \right)$$

$$\mathbf{A_2} = \left(\frac{\partial T_2}{\partial \mathbf{y}} \bigg|_{\hat{\mathbf{y}}} \right) \quad \mathbf{B_2} = \left(\frac{\partial T_2}{\partial \mathbf{u_d}} \bigg|_{\hat{\mathbf{u}}_\mathbf{d}} \right)$$

$$\mathbf{C_2} = \left(\frac{\partial T_2}{\partial \mathbf{u_p}} \bigg|_{\boldsymbol{\mu}_{\mathbf{u_p}}} \right)$$

$$\mathbf{d_2} = -\left(\frac{\partial T_2}{\partial \mathbf{u_d}} \bigg|_{\hat{\mathbf{u}}_\mathbf{d}} \hat{\mathbf{u}}_\mathbf{d} + \frac{\partial T_2}{\partial \mathbf{u_p}} \bigg|_{\boldsymbol{\mu}_{\mathbf{u_p}}} \boldsymbol{\mu}_{\mathbf{u_p}} + \frac{\partial T_2}{\partial \mathbf{y}} \bigg|_{\hat{\mathbf{y}}} \hat{\mathbf{y}} \right)$$

Step 2: Identify the Random Variables and their Distributions

All of the random variables in this example are associated with the parameters and not with the model itself. As shown in Table 2, the values are given by $\sigma_y \sim \mathcal{N}(250 \times 10^3, 625 \times 10^6)$, $\rho \sim \mathcal{N}(7850, 100)$, and $f \sim \mathcal{N}(3.5, 0.49)$.

Step 3: Form the Iterative Equations

In order to use the Kalman filter to simultaneously estimate robustness and converge the design, the iterative equations described in Eq. (20) for fixed-point iteration need to be formed. Through analogy of variables, the matrices are given by

$$\boldsymbol{\Lambda} = \begin{pmatrix} \mathbf{A_1} \\ \mathbf{A_2} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \frac{\partial T_1}{\partial \mathbf{y_2}} \Big|_{\hat{\mathbf{y}}} \\ \frac{\partial T_2}{\partial \mathbf{y_1}} \Big|_{\hat{\mathbf{y}}} & \mathbf{0} \end{pmatrix}$$

$$\boldsymbol{\beta} = \begin{pmatrix} \mathbf{B_1} \\ \mathbf{B_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial T_1}{\partial \mathbf{u_d}} \Big|_{\hat{\mathbf{u}}_\mathbf{d}} \\ \frac{\partial T_2}{\partial \mathbf{u_d}} \Big|_{\hat{\mathbf{u}}_\mathbf{d}} \end{pmatrix}$$

$$\boldsymbol{\gamma} = \begin{pmatrix} \mathbf{C_1} \\ \mathbf{C_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial T_1}{\partial \mathbf{u_p}} \Big|_{\boldsymbol{\mu}_{\mathbf{u_p}}} \\ \frac{\partial T_2}{\partial \mathbf{u_p}} \Big|_{\boldsymbol{\mu}_{\mathbf{u_p}}} \end{pmatrix}$$

$$\boldsymbol{\delta} = \begin{pmatrix} \mathbf{d_1} \\ \mathbf{d_2} \end{pmatrix} = \begin{pmatrix} -\left(\frac{\partial T_1}{\partial \mathbf{u_d}} \Big|_{\hat{\mathbf{u}}_\mathbf{d}} \hat{\mathbf{u}}_\mathbf{d} + \frac{\partial T_1}{\partial \mathbf{u_p}} \Big|_{\boldsymbol{\mu}_{\mathbf{u_p}}} \boldsymbol{\mu}_{\mathbf{u_p}} + \frac{\partial T_1}{\partial \mathbf{y}} \Big|_{\hat{\mathbf{y}}} \hat{\mathbf{y}} \right) \\ -\left(\frac{\partial T_2}{\partial \mathbf{u_d}} \Big|_{\hat{\mathbf{u}}_\mathbf{d}} \hat{\mathbf{u}}_\mathbf{d} + \frac{\partial T_2}{\partial \mathbf{u_p}} \Big|_{\boldsymbol{\mu}_{\mathbf{u_p}}} \boldsymbol{\mu}_{\mathbf{u_p}} + \frac{\partial T_2}{\partial \mathbf{y}} \Big|_{\hat{\mathbf{y}}} \hat{\mathbf{y}} \right) \end{pmatrix}$$

where the numerical values for each of these matrices is evaluated at each subsequent iteration at the appropriate nominal values.

Step 4: Estimate the Mean Output and the Covariance

The equations formed in the prior step can then be propagated through the Kalman filter defined by Eqs. (4)-(10) with

$$\mathbf{F}_{k-1} = \boldsymbol{\Lambda}, \quad \forall k \in \{1, 2, ...\}$$

$$\mathbf{B}_{k-1} = \begin{pmatrix} \boldsymbol{\beta} & \boldsymbol{\gamma} & \mathbf{I_{4\times 4}} \end{pmatrix}, \quad \forall k \in \{1, 2, ...\}$$

$$\mathbf{u}_{k-1} = \begin{pmatrix} \mathbf{u_d} \\ \mathbf{u_p} \\ \boldsymbol{\delta} \end{pmatrix}, \quad \forall k \in \{1, 2, ...\}$$

where in this example $\mathbf{u_d} = \begin{pmatrix} E & l & r_1 & r_2 & g & y_2 & y_3 \end{pmatrix}^T$ and $\mathbf{u_p} = \mathbb{E}(\mathbf{u_p}) = \boldsymbol{\mu}_{\mathbf{u_p}}$. In this example, the matrix $\mathbf{Q}$ is the null matrix since the only uncertain parameters of the problem are associated with the input parameters, not the model. The unscented transform is used on an uncoupled system with the distribution described in Step 2 in order to identify $\mathbf{y_0}$ and $\mathbf{\Sigma_0}$, the initial output mean and covariance for each design. A design is considered converged when the absolute difference between iteration estimates is less than 1×10^{-4} or the relative difference is less than 1×10^{-6}.

Step 5: Identify the Mean and Variance Bound of the Objective Function

Upon convergence, the value of $\mathbf{y}$, the state variable in the problem, is the mean response for each of the components of the output CAs. In this example since the objective is the weight of truss $W_1 + W_2$, the two elements of the second CA output, the response is given by

$$\bar{r} = \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} \hat{\mathbf{y}}_{n|n}$$

The estimate for the variance (*i.e.*, the variance bound) in this case is two times the two-norm of the entire estimated covariance matrix

$$\sigma_r^2 \leq \sum_{i=1}^{2} \parallel \mathbf{\Sigma}_{\mathbf{y^*}_{n|n}} \parallel_2 = 2 \parallel \mathbf{\Sigma}_{\mathbf{y^*}_{n|n}} \parallel_2$$

.

IX.B. Results

The variation in terms of mean and variance for this problem is shown in Fig. 7. From this figure, the

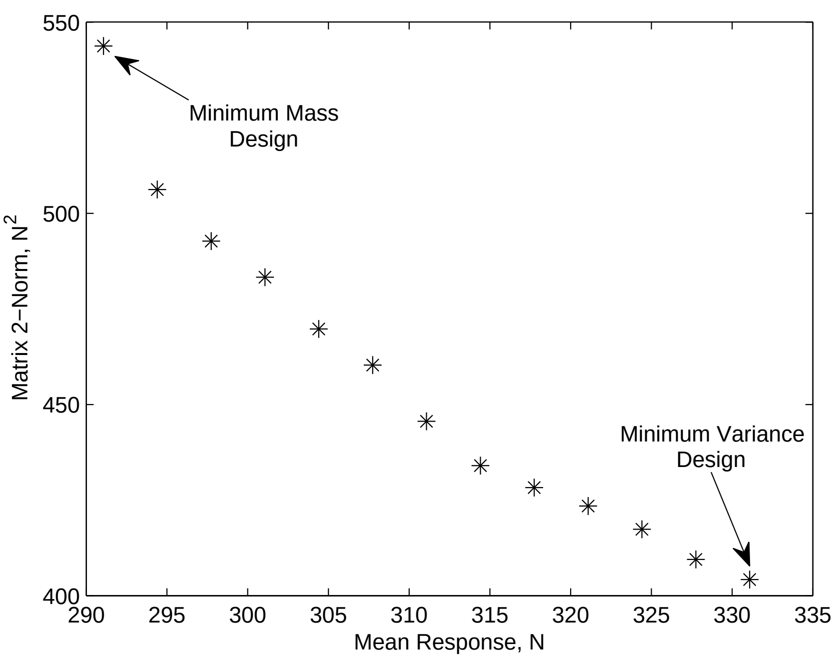

Figure 7. Variation of $2||\mathbf{\Sigma}_{\mathbf{y^*}}||_2$ with the mean objective function for the design of a two bar truss with a load at the mutual joint.

deterministic optimum is the minimum mean solution; however, it is not the minimum variance solution. This minimum variance design is approximately 39 N heavier.

X. Conservatism of the Matrix Two-Norm

An asymptotic error of approximately 40% was observed when sweeping the design space of a linear design with two CAs in the first example application. Each of the CAs had two output variables (*i.e.*, $\dim(\mathbf{y}) = 4$) and there were two probabilistic inputs into the design. For linear systems, the Kalman filter propagates the uncertainty exactly, therefore, this error is a function of the matrix two-norm being used to obtain a bound on the variance. This section quantifies this error as a function of the geometry of the matrix two-norm. The matrix two-norm is defined as

$$\parallel \mathbf{A} \parallel_2 = \max_{\|\mathbf{x}\|_2=1} \parallel \mathbf{Ax} \parallel_2 = \sqrt{\lambda_{\max}(\mathbf{A}^H\mathbf{A})}$$

which, in practical terms means that the matrix two-norm returns a value equal to the maximum variance of the design.

Geometrically, consider the geometry shown in Fig. 8 for the matrix two-norm where x and y are the projection of $\sigma_{X_1'}$ on the σ_{X_1} axis and $\sigma_{X_2'}$ on the σ_{X_2} axis, respectively.

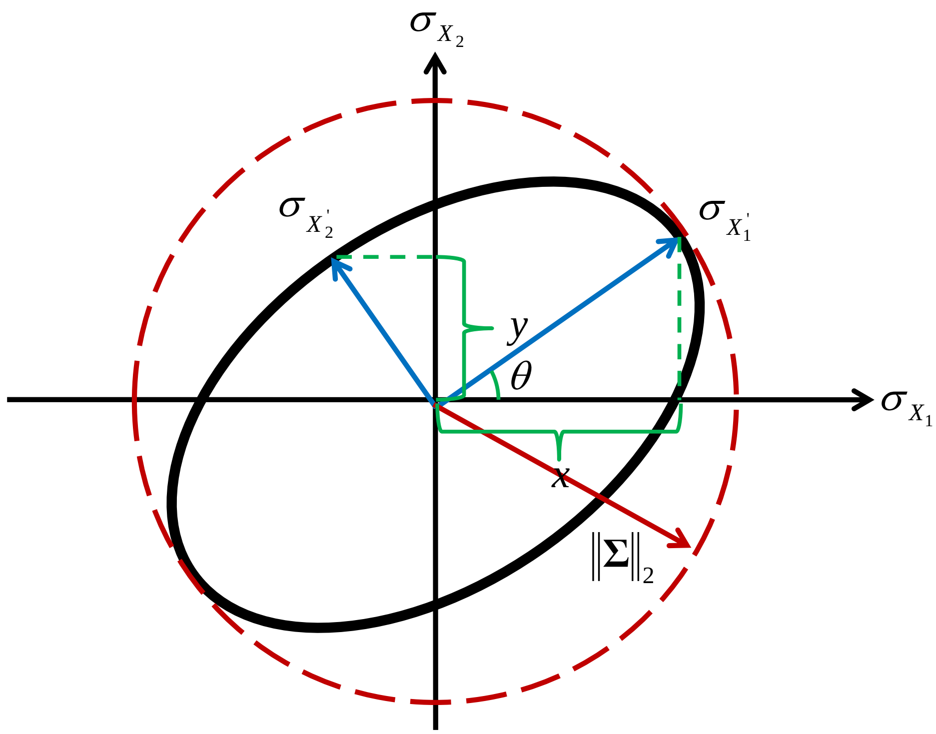

Figure 8. Two-dimensional geometry associated with the matrix two-norm.

In this figure, the matrix two-norm approximates σ_{X_1} with $\sigma_{X_1'}$. The error in this approximation is a maximum when $\theta = 45°$ ($i.e.$, $\cos^{-1}(\hat{\boldsymbol{\sigma}}^T_{X_1}\hat{\boldsymbol{\sigma}}_{X_1'}) = 45°$). The percent error due to this approximation is given by

$$\epsilon_{\%} = 100\left(\frac{\sigma_{X_1'} - x}{x}\right) = 100\left(\frac{\sigma_{X_1'}}{x} - 1\right)$$

where by geometry

$$x = \sigma_{X_1'}\cos\theta$$

Therefore, the expression for error can be reduced to

$$\epsilon_{\%} = 100\left(\frac{1}{\cos\theta} - 1\right)$$

When substituting in $\theta = 45°$, this yields a percent error of 41.42% in the estimation of the standard deviation for the two-dimensional problem of Fig. 8. This result can be generalized to accommodate growth in the dimensionality of the covariance matrix. As the dimensionality of the covariance matrix increases, the value of $\epsilon_{\%}$ decreases due to changes in the geometry of the design space changing θ. This is shown in Fig. 9.

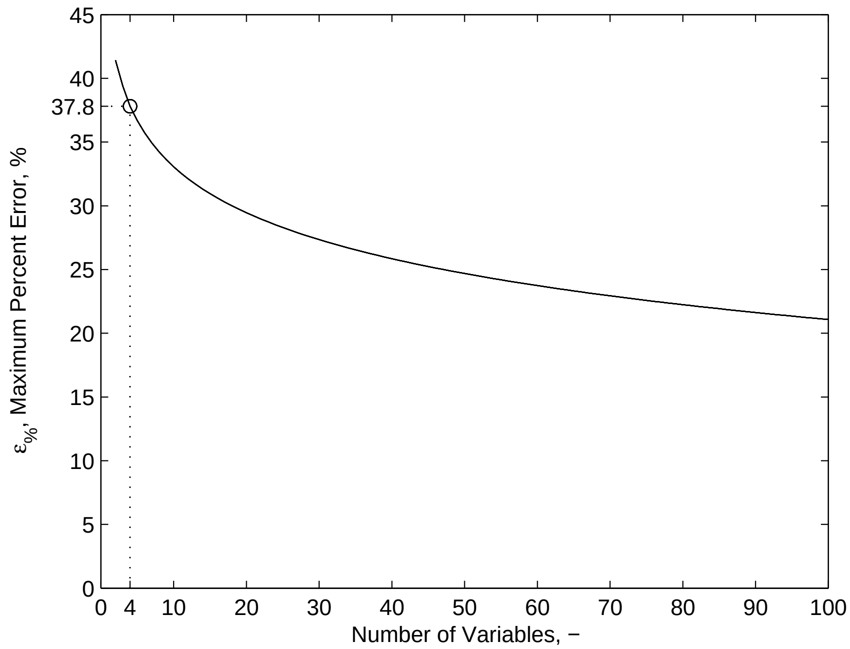

Figure 9. Maximum percent error due to the matrix two-norm approximation as a function of the dimensionality of the problem.

In Fig. 9, for the case of four CA outputs, as was used in the first example problem, this error produces a 37.8% error, which is the error observed in the parameter sweep in Figs. 2-4. Even with a large number of design variables, the two-norm approximation to the standard deviation produces a $20-25\%$ conservative approximation. However, this will only be achieved in cases where there is loose coupling between CAs.

XI. Conclusions

Through viewing the multidisciplinary design problem as a dynamical system a host of tools became available to the MDA/O community. One of these tools is the use of estimation theory. This work demonstrated the applicability of applying the Kalman filter in a manner similar to linear covariance analysis to the multidisciplinary design problem. Estimation techniques were demonstrated through two example problems, one which examined the accuracy of the mean and variance propagation and one which examined the mean and variance for the design of a two-bar truss. It was observed for the linear, two contributing analysis system that the two-norm provided an error less than 37.8% on the standard deviation and less than 0.08% on the mean across a wide-range of problems. This error was shown to be a function of the geometry of the covariance matrix. The use of estimation theory was also observed to be applicable for nonlinear designs through the two-bar truss problem through successive linearization.

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